Mathematics 102
Mathematical Modelling and PreCalculus

Lab Manual

Department of Mathematics & Statistics
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3rd Edition

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Published by the University of Regina Department of Mathematics & Statistics

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Unit 1. Introduction to Mathematical Models

1. A bathtub initially contains 50 litres of water. Water is added at a constant rate of 8 litres per minute.
   a) Define appropriate variables and write an equation to express the total amount of water in terms of time passed.
   b) Use the model to determine how full the bathtub will be in six minutes.
   c) Use the model to determine when the bathtub will contain 200 litres of water.

2. A cell phone plan has a $5.00 monthly flat fee, plus an additional 14 cents per minute of airtime.
   a) Define appropriate variables and write an equation to express the monthly cost in terms of the airtime used.
   b) What will your monthly bill be if you used 83 minutes of airtime?
   c) How much airtime can you get with a monthly budget of $20?

3. A real estate agent charges a flat fee of $4000 plus a commission of 4% of the selling price of a house.
   a) Let F be the total fee that the agent charges. Let p be the selling price of the house. Express F as an equation in terms of p.
   b) The agent sells a house for $225,000. What is her total fee?
   c) If the agent collects a fee of $11,360, how much did the house sell for?
   d) If the agents total fee comes to 6% of the house price, how much did the house sell for?

4. A retailer buys items at price W from the wholesaler, and sells them at a retail price R with a 40% markup.
   a) Express R as an equation in terms of W.
   b) If the wholesale price is $140, what is the retail price?
   c) If the retail price is $210, what is the wholesale price?

5. An elevator begins its journey at a height of 34 m and descends at a rate of 0.6 metres per second. Setup a model for current height in terms of time, and use it to find the time it takes for the elevator to reach the height of 10m.

6. A water barrel initially contains S litres of water. Water leaks out of the barrel at a rate of x litres of water per hour. Let W be the amount of water in the barrel after t hours have passed.
   a) Find an equation for the amount of water that is left in the barrel after t hours, i.e. W in terms of x and t.
   b) If S=200 and x=4, what is the value of W after t=6 hours?
   c) If S=300 and the barrel is empty after 12 hours, what is the value of x?
   d) If x=7 and W=100 after t=9 hours, find the value of S.

Unit 1 - Solutions

1.

a) Let W be the amount of water (in litres). Let t be the time passed (in minutes).

Then Water = Starting Amount + Water added over time

= 50 litres + 8 litres for every minute (i.e. multiplied by t).

Using our variables:

$$W = 50 + 8t$$

b) Substitute t=6 into the equation $W = 50 + 8t$.

Let t=6, then $W = 50 + (8)(6)$

$= 98$

In six minutes, the bathtub will contain 98 litres.

c) Now substitute W=200 into the equation, and solve

$200=50+8t$ for the time t.

Solve to get $150 = 8t$,

i.e. $t = 18\,{}^{3}/_{4}$

It will take 18.75 minutes (or 18 minutes and 45 seconds) for the bathtub to contain 200 litres of water.

2.

a) Let C be the total monthly cost (in \$), let t be the airtime used (in minutes).

Then Total Cost = Flat Fee + Per minute Fee

$$C = 5 + 0.14t$$

b) Substitute time t=83 minutes. Then $C = 5 + (0.14)(83) = \$16.62$

Your monthly bill will be \$16.62.

c)Substitute total cost C=20 and then solve $20 = 5 + 0.14t$ for time t.

Solve to get $15 = 0.14t$

i.e. $t = 107.142...$

You would be able to get 107 minutes of airtime with a budget of \$20.

(note that you have to round down to the next lower integer, as you don't have enough money to cover the 108th minute).

3.

a) The total fee F is made up of the flat free (\$4000) plus the commission (4% of the selling price, i.e. 4% of p). To express the percentage, recall that "percent" literally means "per hundred", so 4% is equivalent to the fraction 4/100 or as a decimal 0.04.

Hence the total fee is $F = 4000 + 0.04p$

b) Let p=225,000, then the total fee will be $F = 4000 + (0.04)(225000)$
$= \$13,000$

The real estate agent's fee for selling a \$225,000 house is \$13,000.

c) Now substitute the fee F = 11360 and solve for the selling price p:

$$11360 = 4000 + 0.04p$$

Solve for p=184,000

The house sold for \$184,000.

d) What do we know: the total fee F is 6% of the total house price,

i.e. expressed in terms of our variables we have

$$F = .06\ p$$

We also know that the fee can always be calculated as

$$F = 4000 + 0.04\ p.$$

We have two equations for F. We can set these equal to each other to solve for the house price p:

$$.06\ p = 4000 + 0.04\ p$$

Solve for $p = 200,000$

The house sold for \$200,000.

Note: with a more complex problem like this one, it may be worth doing a check after the fact. Let's calculate the fee F for a \$200,000 house:

$$F = 4000 + (0.04)(200000)$$
$$= 12000$$

What percentage is \$12,000 out of \$200,000?

$$\frac{12,000}{200,000} \times 100 = 6$$ i.e. 6%, as required.

4.

a) We need to create an equation that gives us the retail price R in terms of the wholesale price W. Here are two ways to think about this:

i) Since the retail price consists of the wholesale price (W) and a 40% markup (0.4 W), the total retail price is $R = W + 0.4\ W$

i.e. $R = 1.4\ W$

ii) Alternatively, since we are saying that the retail price is 140% of the wholesale price, after the markup, we can simply express 140% as a decimal 140/100 = 1.4.

Again, we get $R = 1.4\ W$

b) Let $W = 140$ and calculate R:

$R = (1.4)(140) = 196$. The retail price is \$196.

c) Let $R = 210$ and solve for W:

Solve $210 = 1.4\ W$ for $W = 150$. The wholesale price is \$150.

5. Proceed as in Question #1 or #2.

First define some variables (your choice of letters may be different, of course):

H - current height in metres

t - time passed in seconds

Then the height is given by Height = Initial Height - rate × time

$$H = 34 - 0.6\,t$$

We now have the model. In this case we are supposed to find the time t at which the height is 10 metres. Hence substitute H=10 and solve for t:

$$10 = 34 - 0.6\,t$$

Solve for $t = 40$

It will take 40 seconds for the elevator to reach the height of 10 metres.

6.

a) Note that this model contains four variables:

W - amount of water currently in the barrel
S - initial amount of water in barrel
x - per hour rate at which the barrel leaks.
t - number of hours passed.

The structure of the model is very similar to question #5 (starting position with constant decrease). However, instead of numerical values for starting position and rate, we are simply keeping the variable S and x in place.

The current amount of water (W) is the initial amount (S) minus the leaked amount (x litres per hour, i.e. x times t).
Put this all together into and equation, we have the model $W = S - xt$

b) Now we are given specific values. If the starting amount is $S=200$ litres and the rate is $x=4$ litres/hour, then the current amount of water after 6 hours is $W = 200 - (4)(6) = 176$ litres.

c) Now the initial amount is given as $S=300$, and the barrel is empty ($W=0$) when $t=12$hours.
Insert these values into $W = S - xt$ and solve for the rate x:

$$0 = 300 - 12x$$

Solve for $x=25$. The water leaks at a rate of 25 litres per hour.

d) Insert the given values to get $100 = S - (7)(9)$.
Solve for $S=163$.

What does this tell us? If a barrel leaks at a rate of 7 litres/hour for 9 hours and now contains 100 litres, it must have originally contained 163 litres.

Unit 2. Elementary Algebra: Expressions

1. Simplify each given fraction:

a) $\frac{3x}{8}+\frac{x}{8}$ b) $\frac{2}{5}-\frac{3}{10}$ c) $\frac{2x}{7}+\frac{4x}{5}$

d) $\frac{2}{3}\times\frac{4}{5}$ e) $\frac{2}{3}\div\frac{4}{5}$ f) $6\div\frac{1}{2x}$

g) $\frac{x}{2}\div 6$ h) $\left(\frac{8}{18}-\frac{2}{9}\right)\div\frac{3}{9}$ i) $\left(\frac{3}{5}\times\frac{3}{4}\right)-\left(\frac{9}{4}\times\frac{1}{3}\right)$

j) $\left(\frac{4}{9}\times\frac{3}{8}\right)+\left(\frac{5}{4}\div\frac{3}{8}\right)$ k) $3\left(\frac{x}{2}+\frac{2x}{3}\right)-\frac{5x}{6}$ l) $\frac{3}{x}+\frac{2}{y}-\frac{1}{2}\div\frac{x}{5}$

2. Simplify:

a) $2^2 2^3$ b) $x^5(x^3)^2$ c) $\frac{y^5}{y^2}$

d) $\left(\frac{z^3z^6}{z^4}\right)^2$ e) $\left(\frac{2}{5}\right)^{-3}$ f) $\left(\frac{3}{7}\right)^0$

g) $x^{-1/4}x^{3/4}$ h) $\left(\frac{2x^6y^{-2}}{5x^3y^4}\right)^{-2}$ i) $\left(4x^{-3}y^{-4}\right)\left(-3x^3y^{-3}\right)^{-2}$

j) $\frac{\left(-4xy^3\right)^3}{\left(-2x^7y\right)^4}$ k) $\frac{\left(5x^{-3}y^2\right)^3(2xy)^{-1}}{(3x^2y^{-1})^2}$ l) $\left(\frac{8^{-1}x^{-2}y}{2x^4y^{-1}}\right)^{-2}$

3. Simplify:

a) $\sqrt{36x^3y^4}$ b) $\sqrt[4]{\frac{2x^8y^2}{32x^4y^6}}$ c) $\left(\frac{-8x^2y^8}{x^5y^2}\right)^{\frac{1}{3}}$

d) $16^{\frac{3}{4}}$ e) $\left(\frac{8}{27}\right)^{-2/3}$ f) $3\sqrt{2x}-5\sqrt{2x}$

g) $4\sqrt{3}+2\sqrt{27}$ h) $2\sqrt[3]{54x}-\sqrt[3]{16x}$ i) $\left(\frac{81x^4}{\sqrt[3]{4096y^6}}\right)^{-1/4}$

4. Rationalize the denominator:

a) $\dfrac{2}{\sqrt{5}}$ b) $\sqrt{\dfrac{7}{3x}}$ c) $\dfrac{3}{2+\sqrt{x}}$ d) $\dfrac{2+3\sqrt{7}}{5-2\sqrt{7}}$

5. Factor completely:

a) $4x^3+2x$ b) $x^2+12x+20$ c) x^3-5x^2+6x

d) $16x^5-x$ e) $(xy-2x)+(5y-10)$ f) $\sqrt{x}-\sqrt{x^3}$

g) a^2-6a+9 h) $6-7x-5x^2$ i) $10a^2-29a+10$

j) x^3+x^2-4x-4 k) $2x^3-11x^2-40x$ l) $x(x^2-2x)+x-2$

m) $3xy^3+10xy^2+3xy$ n) $x^{7/4}+2x^{3/4}+x^{-1/4}$ o) $\dfrac{3}{7}(x+1)^{\frac{7}{3}}-\dfrac{3}{4}(x+1)^{\frac{4}{3}}$

6. Simplify:

a) $\dfrac{x^2+3x+2}{x^2-1}$ b) $\dfrac{4-9x^2}{3x^2+x-2}$ c) $\dfrac{2x^2-x-1}{2x^2+7x+3}$

d) $\dfrac{\frac{1}{x}-\frac{1}{y}}{x-y}$ e) $\dfrac{(x-2)^{\frac{5}{2}}-(x-2)^{\frac{3}{2}}}{x^2-5x+6}$ f) $\dfrac{2x^6y-5x^5y+3x^4y}{2x^8y^3-x^7y^3-3x^6y^3}$

7. Simplify:

a) $\dfrac{4x^2-4y^2}{6x^2y^2}\div\dfrac{3x^2+3xy}{2x^2y-2xy^2}$

b) $\dfrac{x-2}{x+1}-\dfrac{3-12x}{2x^2-x-3}$

c) $\dfrac{3}{2x-3}+\dfrac{2x}{3-2x}$

Unit 2 - Solutions

1.

a) $\frac{x}{2}$ b) $\frac{1}{10}$ c) $\frac{38x}{35}$ d) $\frac{8}{15}$ e) $\frac{5}{6}$ f) $12x$

g) $\frac{x}{12}$ h) $\frac{2}{3}$ i) $-\frac{3}{10}$ j) $\frac{7}{2}$ k) $\frac{8x}{3}$ l) $\frac{4x+y}{2xy}$

2.

a) 32 b) x^{11} c) y^{3} d) z^{10} e) $\frac{125}{8}$ f) 1

g) $x^{1/2}$ h) $\frac{25y^{12}}{4x^{6}}$ i) $\frac{4y^{2}}{9x^{9}}$ j) $\frac{-4y^{5}}{x^{25}}$ k) $\frac{125y^{7}}{18x^{14}}$ l) $\frac{256x^{12}}{y^{4}}$

3.

a) $6x^{3/2}y^{2}$ b) $\frac{x}{2y}$ c) $\frac{-2y^{2}}{x}$ d) 8 e) $\frac{9}{4}$ f) $-2\sqrt{2x}$

g) $10\sqrt{3}$ h) $4\sqrt[3]{2x}$ i) $\frac{2\sqrt{y}}{3x}$

4.

a) $\frac{2\sqrt{5}}{5}$ b) $\frac{\sqrt{21x}}{3x}$ c) $\frac{6-3\sqrt{x}}{4-x}$ d) $-\frac{52+19\sqrt{7}}{3}$

5.

a) $2x(2x^{2}+1)$ b) $(x+10)(x+2)$ c) $x(x-2)(x-3)$

d) $x(2x-1)(2x+1)(4x^{2}+1)$ e) $(y-2)(x+5)$ f) $\sqrt{x}(1-x)$

g) $(a-3)^{2}$ h) $-(5x-3)(x+2)$ i) $(2a-5)(5a-2)$

j) $(x-2)(x+2)(x+1)$ k) $x(x-8)(2x+5)$ l) $(x-2)(x^{2}+1)$

m) $xy(y+3)(3y+1)$ n) $x^{-1/4}(x+1)^{2}$ o) $(x+1)^{4/3}(\frac{3}{7}x-\frac{9}{28})$

6.

a) $\frac{x+2}{x-1}$ b) $-\frac{2+3x}{x+1}$ c) $\frac{x-1}{x+3}$ d) $\frac{-1}{xy}$ e) $(x-2)^{1/2}$ f) $\frac{(x-1)}{(x+1)x^{2}y^{2}}$

7.

a) $\frac{4(x-y)^{2}}{9x^{2}y}$ b) $\frac{2x+3}{2x-3}$ c) -1

Unit 3. Elementary Algebra: Equations

1. Solve each equation:

a) $x+5=14-\frac{x}{2}$ b) $\frac{2x}{3}+\frac{1}{2}=1$ c) $\frac{x+1}{3}=1-\frac{x+5}{2}$

d) $x^2+2x=-1$ e) $6x^2=x+1$ f) $2(x+1)^2-24=0$

g) $x-9=x(x-5)$ h) $x^5-2x^3-3x=0$ i) $4x^4+4x+1=0$

j) $\frac{4}{x+5}=\frac{1}{2x+3}$ k) $\frac{3}{x}-\frac{2}{2x-1}=1$ l) $5+\frac{8}{x-2}=\frac{4x}{x-2}$

m) $4x^4+8x^3-5x^2=0$ n) $x^4-6x^2+5=0$ o) $2x^5=5x^3+3x$

2. Solve each equation by using the quadratic formula:

a) $3x^2+x-1=0$ b) $x^2-4x-30=0$ c) $2x^2+7x=2$

d) $3x^2+x+1=0$ e) $9x^2+169=78x$ f) $x^4-3x^3+x^2=0$

3. Complete the square to write the equation in the form $a(x-b)^2+c=0$, then solve the equation.

a) $x^2+6x-7=0$

b) $x^2-9x+20=0$

c) $4x^2-4x-1=0$

d) $\frac{1}{2}x^2-3x+2=0$

Unit 3 - Solutions

1.

a) $x = 6$

b) $x = \frac{3}{4}$

c) $x = -\frac{11}{5}$

d) $x = -1$

e) $x = \frac{1}{2}, \quad x = -\frac{1}{3}$

f) $x = -1 \pm 2\sqrt{3}$

g) $x = 3$

h) $x = 0, \quad x = \pm\sqrt{3}$

i) no solution

j) $x = -1$

k) $x = 1, \quad x = \frac{3}{2}$

l) no solution

m) $x = 0, x = -\frac{5}{2}, x = \frac{1}{2}$

n) $x = \pm 1, \quad x = \pm\sqrt{5}$

o) $x = 0, \quad x = \pm\sqrt{3}$

2.

a) $x = \frac{-1 \pm \sqrt{13}}{6}$

b) $x = 2 \pm \sqrt{34}$

c) $x = \frac{-7 \pm \sqrt{65}}{4}$

d) no solution

e) $x = \frac{13}{3}$

f) $x = 0, \quad x = \frac{3 \pm \sqrt{5}}{2}$

3.

a) $(x+3)^2 - 16 = 0$, solution $x = 1$, $x = -7$

b) $(x - \frac{9}{2})^2 - \frac{1}{4} = 0$, solution $x = 4, \quad x = 5$

c) $4(x - \frac{1}{2})^2 - 2 = 0$, solution $x = \frac{1}{2} \pm \frac{\sqrt{2}}{2}$

d) $\frac{1}{2}(x-3)^2 - \frac{5}{2} = 0$, solution $x = 3 \pm \sqrt{5}$

Unit 4. Models involving Geometric Shapes

1. A rectangle is three times as wide as it is long. Let A be the area of the rectangle. Let P be the perimeter of the rectangle. Let x and y be the length and width of the rectangle, respectively.

    a) Write "the width is three times the length" as an equation.
    b) Express the area as an equation in terms of the length. If the length of the rectangle is 4 cm, what is its area?
    c) Express the perimeter as an equation in terms of the length. If the perimeter is 16 cm, what is the length?
    d) Express the area in terms of the perimeter. If the perimeter is 8 cm, what is the area?

2. A rectangular parking lot is 20 metres longer than it is wide. If its total area is 10,304 m^2, what are its dimensions?

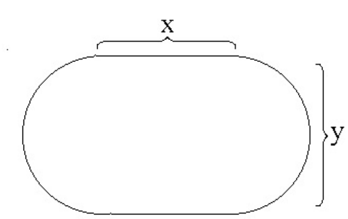
3. A rectangular picture is mounted in a one inch thick frame, that is, the picture is surrounded by one inch of framing on all four sides. The frame is two inches wider than it is long. If the area of the frame (including the picture) is 168 in^2, what is the area of just the picture?

4. A closed box with square bottom is three times as high as wide.
    a) Express the surface area in terms of width.
    b) Express the volume as in terms of width.
    c) Express the surface area in terms of volume.
    d) A box as volume $24m^3$. What is its surface area?

5. A rectangular box has a square bottom and is 4 cm taller than it is wide.
    a) Find the volume (V) of the box expressed in terms of its width (x).
    b) Find the surface area (A) of the box expressed in terms of its width (x).
    c) If the box has a width of 7 cm, what is its volume and surface area?

6. Express the perimeter of a circle in terms of its area, i.e. find an equation for P in terms of A. Then use this model to find the perimeter of a circle with area 4π.

7. A track'n'field stadium is basically a rectangle with two semicircles at either end.



    a) Assuming that the length "x" of the stadium is 100m, express the perimeter in terms of its width "y".
    b) Assuming that the perimeter of the stadium is 500m, express the area in terms of width "y".
    c) If the stadium has perimeter 500m and width 50m, find its area.

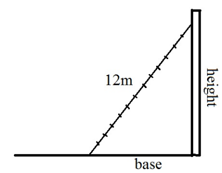
8. If a 25m tall tree casts a 15m long shadow, how tall is a tree that casts an 18m long shadow?

9. A building casts a 42m long shadow. A 2^{nd} building is 8m taller and casts a 46m long shadow. How tall is the first building?

10. A person is standing 18m away from a 10m tall light post. If the person is 1.70m tall, how long of a shadow does (s)he cast?

11. A bee leaves her nest, flies 15m straight east, then 20m straight south, then flies directly back to her nest. What was the total distance of her flight?

12. A 12m long ladder rests against a wall. The distance from the top of the ladder to the ground (the "height") is 3m longer than the distance from the base of the ladder to the wall (the "base"). What is the distance from the base of the ladder to the wall?



Unit 4 - Solutions

1.

a) Width equals three times length, i.e. $y = 3x$.

b) Area is length times width, i.e. $A = xy$

We need to express A in terms of x only only, hence replace y with an expression in terms of x. Use your answer from (a), $y=3x$.

Then $A = x\,(3x) = 3x^2$.

Now, if length $x=4$ cm, then area $A=3(4)^2 = 48\text{ cm}^2$.

c) The perimeter is the sum of all four sides: 2 lengths and 2 widths,

i.e. $P = 2x + 2y$.

Again, replace x in terms of y,

to get $P = 2x + 2\,(3x)$

$= 2x + 6x$

$= 8x$

If perimeter $P = 16$ cm, then solve $16 = 8x$ for length $x=2$ cm.

d) Express A in terms of P.

We already have an equation A in terms of x: $A = 3x^2$

We also have an equation P in terms of x: $P = 8x$

We can eliminate the common variable x: Solve $P = 8x$ for $x = P/8$

then substitute into the area equation:

$A = 3\,(P/8)^2 = {}^3/_{64}\,P^2$

Given a perimeter of $P = 8$ cm, we get an area of $A = {}^3/_{64}\,(8)^2$

$= 3\text{ cm}^2$

2. Let x be the width of the lot, then x+20 is the length.

The area is of a rectangle is Area = width × length,

i.e. $A = (x)(x+20)$

Since the area is given as $A=10304$, we can solve the equation $10304 = (x)(x+20)$ for x.

Expand to find the quadratic equation $x^2 + 20x - 10304 = 0$.

Solve for $x=92$ (factor or use quadratic formula).

Hence the dimensions are 92 by 112 metres.

3. Let x be the length of the frame, then its width is x+2

and its area is $A = (x)(x+2)$.

We know the frame has area $A=168$, hence solve $168 = (x)(x+2)$

i.e. $x^2 + 2x - 168 =0$

Solve for $x=12$. Hence the frame measure 12 by 14 inches.

Since it is one inch thick, the picture measures 10 by 12 inches, with area 120 in^2.

4. Let x be the width of the box. Then the length is "x" as well, and the height is "3x".

a) The surface area is A = Area of top/bottom + Area of sides

= 2 (Area of bottom) + 4 (Area of side)

$= 2\,(x^2) + 4\,(3x^2)$

$= 14x^2$.

b) The volume is V = length * width * height

$= 3x^3$

c) To express surface area in terms of volume, fir express x in terms of V:

$x = (V/3)^{1/3}$.

Then substitute into the equation for A to get

$A = 14\,(V/3)^{2/3}$

d) Use the equation for A in (c) with V=24 to get a surface area of 56 m^2.

5. Let x be the width (and length) of the box, then x+4 is its height.

a) The volume is given by V = (width)(length)(height)

$= (x)(x)(x+4)$

$= x^3+4x^2$

b) Note that the top/bottom have the same area, as do each of the four sides.

The surface area is then given by

A = 2 (Area of bottom) + 4 (Area of side)

$= 2x^2 + 4\,x(x+4)$

$= 6x^2 + 16x$

c) If the width is x=7 cm, then the box has volume V=539 cm^3 and area A=406 cm^2.

6. Let x be the radius of the circle. We know that for a circle, we have the formulas

Area $A=\pi x^2$ and Perimeter $P=2\pi x$.

We wish to express P in terms of A, hence we need to eliminate the common "x" that appears in both formulas.

First, solve A for x: $A=\pi x^2$ so therefore $x=(A/\pi)^{1/2}$

Now substitute this expression for x in terms of A into the formula for P to get

$P = 2\pi\,(A/\pi)^{1/2}$

$= 2\,(\pi A)^{1/2}$ - we now have an equation for P in terms of A only.

Finally the perimeter of a circle with radius 4π is $P= 2\,(\pi\,(4\pi))^{1/2}$

$= 4\pi$ (coincidentally!).

7. Note that the radius of the semi-circles "r" is simply half the width "y", i.e. r = y/2.

a) The perimeter of the stadium is

$P = 2x$ (straight lines at top and bottom) + $2\pi r$ (two semi-circles at sides)

Since x is given to be 100m, and r = y/2, we have

$P = 200 + 2\pi\,(y/2)$

$= 200 + \pi y$

b) The area of the stadium is

$A = xy$ (the rectangle) $+ \pi r^2$ (the two semi-circles).

We need to replace "x" and "r" in terms of "y"s. First use $r=y/2$ to get

$A = xy + \pi(y/2)^2$

$= xy + \pi y^2/4$ Let call this equation (1).

Now, the perimeter is given as $P = 500$. We also know (see (a)) that

$P = 2x + 2\pi r$

$= 2x + \pi y$

Hence solve $500 = 2x + \pi y$

for $x = (500 - \pi y)/2$

and substitute into equation (1) to get

$A = y(500 - \pi y)/2 + \pi y^2/4$

$= 250\,y - \pi y^2/4$

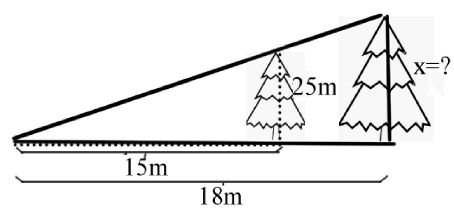
c) Use part(b) (since $P = 500$m) and $y=50$ to get $A(50) = 12500 - 625\pi \approx 10536.5\ m^2$

8. Let x be the height of the second tree.

Note that it must be taller than 25m.

Set up similar triangle ratios: $\dfrac{25}{15} = \dfrac{x}{18}$.

Solve for x=30. The tree is 30m tall.



9. Let x be the height of the first building. Then the second building has height x+8.

Now set up similar triangles: $\dfrac{x}{42} = \dfrac{x+8}{46}$.

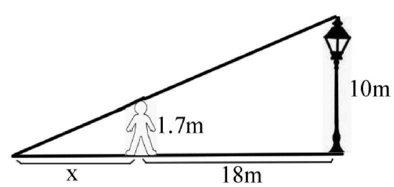
Solve for x=84. The first building is 84m tall.

10. Let x be the length of the person's shadow.

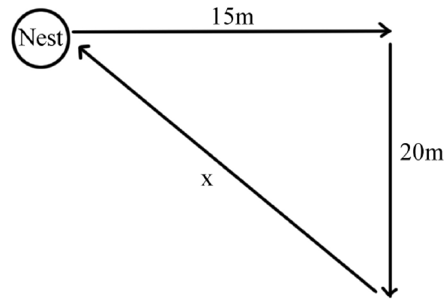
Set up similar triangle ratios, comparing the person (small triangle) and the lamp post (large triangle):

$$\frac{x}{1.7} = \frac{18+x}{10}$$

Solve for x=30.6/8.3. The shadow will be about 3.69m long.



11. The bee's flight path is a right triangle.



Use the pythagorean theorem to find the length of the diagonal: $(15)^2+(20)^2=x^2$.
Solve for x=25.

Hence the total flight was 15m + 20m + 25m = 60m.

12. Let x be the length of the base. We are told that the height is 3m more than the base, i.e. the height is x+3.

Now use the pythagorean theorem: $(x)^2+(x+3)^2=(12)^2$.

Solve for $x = \dfrac{-6 \pm \sqrt{1116}}{4}$.
The base length is approximately x=6.85m.

Unit 5. Further Problem Solving Examples

1. The breaking distance of a car is defined as the distance the car travels from the moment the brakes are applied to the moment the car comes to a complete stop. Suppose the breaking distance of a car, "D" in metres, is one two-hundredths of the square of the car's speed, "s" in km/h.
   a) Express the breaking distance in terms of the the car's speed.
   b) What is the breaking distance of this car traveling at 60 km/h? 90 km/h? 120 km/h?
   c) If the breaking distance of the car was 162 metres, how fast was the car traveling?

2. Two fishing boats depart a harbour at the same time, one traveling east, and the other west. The eastbound ship travels at a speed of 3 km/h faster than the westbound ship. After 2 hours the boats are 30 km apart.
   a) Find the speed of both boats.
   b) Repeat the question if the faster boat travels east and the slower boat travels south.

3. Alex can paint a room in 6 hours. Bob can paint a room in 5 hours. How long would it take them to paint the room if they worked together?

4. On a 250 km drive to Saskatoon I averaged a speed of 90 km/h. On my way back, I averaged 105 km/h. What was my average speed for the entire trip?

5. A bicyclist bikes up a 4 km uphill trail, then turns around and races down the same trail downhill. On the way down, her speed is 30 km/h faster than on the way up. If she completes the entire trip (uphill and downhill) in 16.8 minutes, what was her speed going uphill?

6. Consider three consecutive positive integers. Their product is exactly 2400 times larger than the middle number. What are the three integers?

7. You have twice as many loonies as quarters, and two more nickels than loonies.
   a) Express the number of nickels in terms of the number of quarters.
   b) Express your total wealth in terms of the number of nickels.
   c) You have 4 nickels in your pocket. How much money do you have in total?

Unit 5 - Solutions

1.

a) Breaking distance D = (1/200) s^2

b) If s=60 km/h, the breaking distance is D = (1/200) $(60)^2$
= 18 metres.

If s=90 km/h, the breaking distance is D = (1/200) $(90)^2$
= 40.5 metres.

If s=120 km/h, the breaking distance is D = (1/200) $(120)^2$
= 72 metres.

c) Let D=162 and solve for s:

162 = (1/200) s^2

s^2 = 32400

s = 180 km/h. The car was driving at a speed of 180 km/h.

2. Let x be the speed of slower boat, then x+3 is the speed of the faster boat.

a) Since they travel in opposite direction, their distance is increasing at speed x+(x+3)=2x+3.
After 2 hours they are 30 km apart, i.e. solve 2(2x+3)=30 for x=6.
The slower boat travels at 6 km/h, the faster boat at 9 km/h.

b) Now the boats form a right triangle with sides 2x and 2(x+3) after 2 hours have passed.
Their distance, the hypotenuse, is 30 km, i.e. solve

$$\sqrt{(2x)^2+(2x+6)^2} = 30$$

$$(2x)^2+(2x+6)^2 = 900$$

$$8x^2+24x-864=0$$

Solve for x=9 and x=-12 (discard).
The slower boat travels at 9 km/h, the faster boat at 12 km/h.

3. Let x be the time (in hours). Every hour, Alex paints 1/6 of the room, Bob paints 1/5 of the room. Hence in x hours, we have $\frac{x}{5}+\frac{x}{6}=1$.
Solve for x=30/11, or about 2.73 hours.

4. No variables are required for this problem. On the way there, I took 250/90 ≈2.78 hours. On the way I took 250/105 ≈ 2.38 hours. The total trip was 500 km. The total time was 5.16 hours. Hence my total average speed was 96.9 km/h. (Note: NOT (90+105)/2).

5. Note that 16.8 minutes = 0.28 hours. Let x be the speed uphill. Then $\frac{4}{x}+\frac{4}{x+30}=0.28$.
Solve for x=20. The uphill speed was 20 km/h.

6. Let x be the smallest integer. Then $(x)(x+1)(x+2)=2400(x+1)$.

Note that $(x+1)$ can be cancelled, leaving just $(x)(x+2)=2400$. Solve for $x=48$.

The three integers are 48, 49, and 50.

7. Let L,Q,N be the number of loonies, quarters, and nickels respectively. Let W be your total wealth. The following information is given:

$L = 2Q$ (1) and $N = L + 2$ (2)

a) To find N in terms of Q, use equation (2) and replace the "L"-term. To do so, re-arrange equation (1) to get

$$N= 2Q + 2$$

b) Your total wealth (in cents) is $W = 100L + 25Q + 5N$.

Now use equations (1) and (2) to replace the "L" and "Q" terms with terms involving "N":

$$\begin{aligned} W &= 100\ (N-2) + 25\ (L/2) + 5\ N \\ &= 100\ (N-2) + 25\ (N-2)/2 + 5N \\ &= 117.5\ N - 225 \end{aligned}$$

c) Use W with N=4 to get $W(4) = 245$ i.e. \$2.45.

Unit 6. Absolute Values and Inequalities

1. Solve each inequality:

a) $3x + 5 < 2$ b) $2 - x \geq 5$ c) $2x + 7 < 5x - 1$

d) $\frac{3}{4}x < \frac{1}{2} - \frac{2}{3}x$ e) $\frac{x}{3} \geq 1 - \frac{x}{5}$

2. Solve each double inequality:

a) $-5 < 6x + 1 < 7$ b) $-7 \leq \frac{1}{2}(16 - 5x) < 23$

3. Solve each equation:

a) $|3x - 1| = 8$ b) $-\frac{1}{2}|x - 1| = 4$ c) $|2x - 4| = x + 2$

d) $|1 - x| = 2x + 1$ e) $|x| + x = 3$

4. Solve each inequality:

a) $|x + 3| < 5$ b) $|x - 2| \geq 6$ c) $\sqrt{(2x - 6)^2} \leq 8$

Unit 6 - Solutions

1.

a) Solution $x < -1$ or in interval notation $x \in (-\infty, -1)$

b) Solution $x \le -3$ or in interval notation $x \in (-\infty, -3]$

c) Solution $x > 8/3$ or in interval notation $x \in (\frac{8}{3}, +\infty)$

d) Solution $x < 6/17$ or in interval notation $x \in (-\infty, \frac{6}{17})$

e) Solution $x \ge 15/8$ or in interval notation $x \in [\frac{15}{8}, +\infty)$

2.

a) Solution $-1 < x < 1$ or in interval notation $x \in (-1, 1)$

b) Solution $-6 < x \le 6$ or in interval notation $x \in (-6, 6]$

3.

a) Case I: $3x - 1 = 8$

$x = 3$ Verify $|8| = 8$ ✓

Case II: $-(3x - 1) = 8$

$x = -7/3$ Verify $|-8| = 8$ ✓

Solution: $x = 3, x = -7/3$

b) This simplifies to $|x - 1| = -8$

It is impossible for an absolute value to equal a negative number, hence there is no solution.

c) Case I: $2x - 4 = x + 2$

$x = 6$ Verify $|8| = 8$ ✓

Case II: $-(2x - 4) = x + 2$

$x = 2/3$ Verify $|-8/3| = 8/3$ ✓

Solution: $x = 6, x = 2/3$

d) Case I: $1 - x = 2x + 1$

$x = 0$ Verify $|1| = 1$ ✓

Case II: $-(1 - x) = 2x + 1$

$x = -2$ Verify $|3| = -3$ ×

Only one solution, $x = 0$

e) Case I: $x + x = 3$

$x = 3/2$ Verify $|3/2| + 3/2 = 3$ ✓

Case II: $-x + x = 3$

$0 = 3$ Impossible.

Only one solution, $x = 3/2$

4.

a) Solution $-8 < x < 2$ or in interval notation $x \in (-8, 2)$

b) Solution $x \le -4$ or $x \ge 8$ or in interval notation $(-\infty, 4] \cup [8, \infty)$

c) Solution $-1 \le x \le 7$ or in interval notation $x \in [-1, 7]$

Unit 7. Graphs, Distances, and Circles

1. Test each of the equations to determine if their graphs are symmetric with respect to the x-axis, with respect to the y-axis, with respect to the origin, or any combination of these.

   a) $x^3 - 4y^2 + 1 = 0$ b) $2+4x = 3y - 1$

   c) $x^3y - x^2 = 2$ d) $2x^4 - 81 = 3y^4 - 2x^2y^2$

2.

a) Sketch the graph of the equation $y = 1 - x^2 + 2x$
   using a table of values with x-values -1, 0, 1, 2, and 3.
   Then, based on your graph, estimate the position of any x- and y-intercepts.

b) Sketch the graph of the equation $y = \frac{x}{2} - \frac{x^3}{8}$ by first determining its symmetry and then
   using a table of values with x-values 0, 1, 2, and 3 only.

c) Sketch the graph of the equation $y = \frac{x^2 - x + 1}{4x - 2}$
   using a table of values with x-values -2, -1, 0, 1, and 2.
   Then create a more accurate graph of the equation using either a graphing calculator or an online graphing calculator tool (google: graphing calculator). Compare and comment on the danger of relying only on table of values to create graphs.

3. Find the (Euclidean) distance between the given points:
   a) (x,y) = (3, -1) and (x,y) = (-1, 5)
   b) (x,y) = (2/3, 1/4) and (x,y) = (1, 2/3)

4. State the equation of the following graphs:
   a) A circle with centre (x,y) = (3, -1) and radius 4.
   b) A circle with centre at the origin and radius 1.
   c) A circle that travels through points (-1,1), (3,1) and (1,3)

5. Sketch the graph of each circle by finding the centre and radius:

   a) $(x-1)^2 + y^2 = 4$ b) $x^2 = 9 - y^2$

   c) $(x+1)^2 + (y-2)^2 = 9/4$ d) $x^2 + 2x + y^2 = 0$

6. Sketch the graph of each ellipse:

   a) $4x^2 + y^2 = 1$ b) $9x^2 + 4y^2 = 36$

Unit 7 - Solutions

1.

a) x-axis? Replace y with (-y): $x^3 - 4(-y)^2 + 1 = 0$
$x^3 - 4y^2 + 1 = 0$ → same equation!
y-axis? Replace x with (-x): $(-x)^3 - 4y^2 + 1 = 0$
$-x^3 - 4y^2 + 1 = 0$ → not the same equation!
origin? Replace both x and y: $(-x)^3 - 4(-y)^2 + 1 = 0$
$-x^3 - 4y^2 + 1 = 0$ → not the same equation!
Final Answer: this graph is symmetric with respect to the x-axis only.

b) x-axis? Replace y with (-y): $2+4x = 3(-y) - 1$
$2+4x = -3y - 1$ → not the same equation!
y-axis? Replace x with (-x): $2+4(-x) = 3y - 1$
$2 - 4x = 3y - 1$ → not the same equation!
origin? Replace both x and y: $2+4(-x) = 3(-y) - 1$
$2 - 4x = -3y - 1$ → not the same equation!
Final Answer: this graph does not have any of these symmetries.

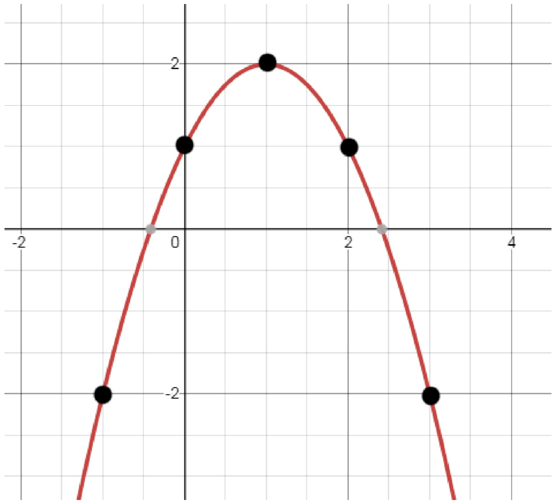
c) x-axis? Replace y with (-y): $x^3(-y) -x^2 = 2$
$-x^3y -x^2 = 2$ → not the same equation!
y-axis? Replace x with (-x): $(-x)^3y -(-x)^2 = 2$
$-x^3y -x^2 = 2$ → not the same equation!
origin? Replace both x and y: $(-x)^3(-y) -(-x)^2 = 2$
$x^3y -x^2 = 2$ → same equation!
Final Answer: this graph is symmetric with respect to the origin only.

d) x-axis? Replace y with (-y): $2x^4 - 81 = 3(-y)^4 - 2x^2(-y)^2$
$2x^4 - 81 = 3y^4 - 2x^2y^2$ → same equation!
y-axis? Replace x with (-x): $2(-x)^4 - 81 = 3y^4 - 2(-x)^2y^2$
$2x^4 - 81 = 3y^4 - 2x^2y^2$ → same equation!
origin? Replace both x and y: $2(-x)^4 - 81 = 3(-y)^4 - 2(-x)^2(-y)^2$
$2x^4 - 81 = 3y^4 - 2x^2y^2$ → same equation!
Final Answer: this graph has all three types of symmetries.

2. a)

x	-1	0	1	2	3
y	-2	1	2	1	-2

Graph:



This appears to be a parabola.

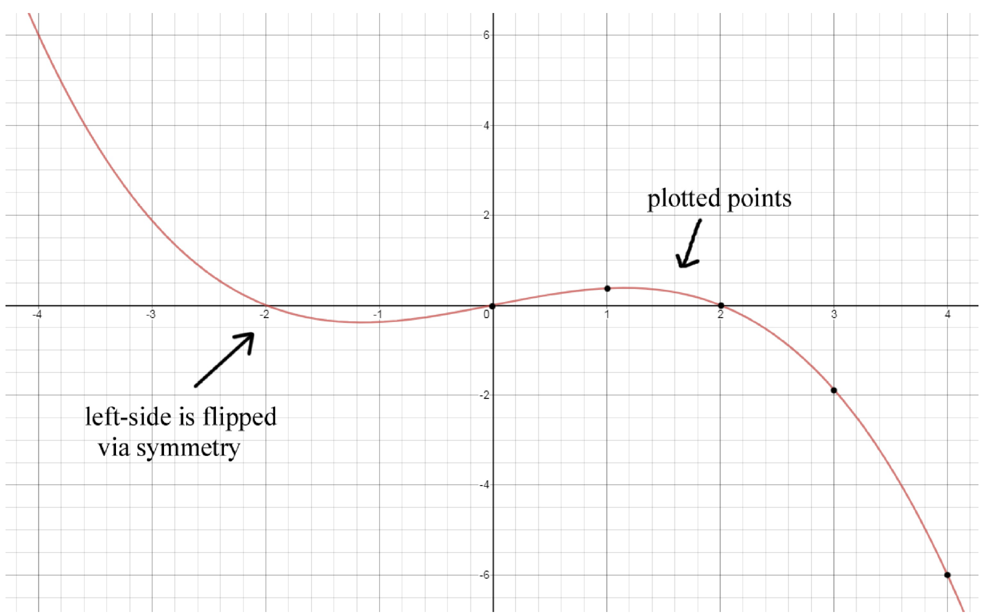
Estimating from the graph,
the y-intercept is at y=1
(we get this value precisely)
the x-intercepts are roughly at x=0.4
and x=2.4.

(Note: setting y=0 and solving for x, we can find the precise values to be y=1±√2)

b) First, we can determine that this equation is symmetrical across the x-axis. Hence we only need to calculate/plot positive x-values. The left half of the graph can then be obtained by flipping the right half.

x	0	1	2	3	4
y	0	0.375	0	-1.875	-6

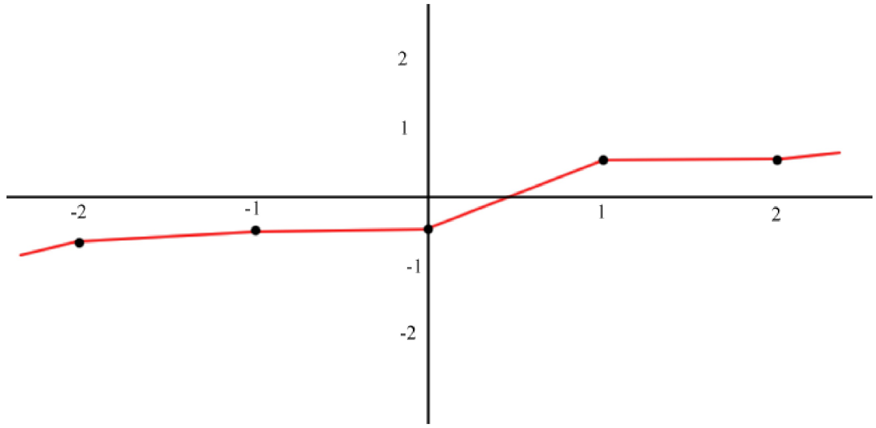
Graph:



c)

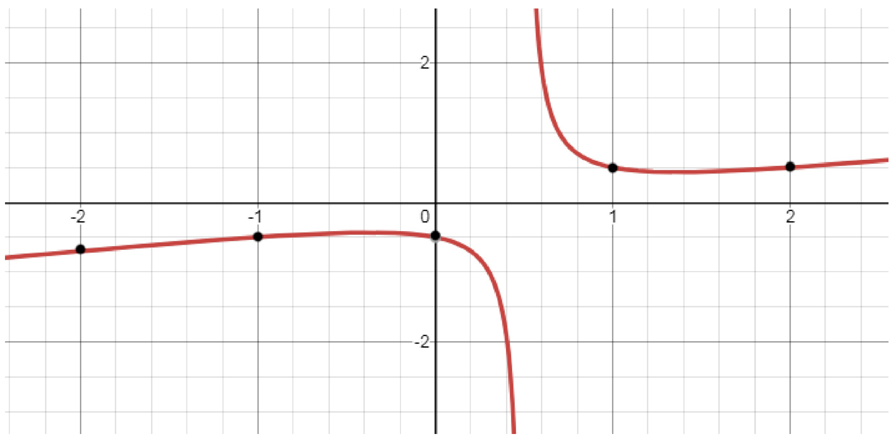
x	-2	-1	0	1	2
y	-0.7	-0.5	-0.5	0.5	0.5

Trying to graph this points from the table of values alone, we might guess that the graph is some sort of "wobbly line":



In reality, however, this graph looks quite different. By picking our particular x-values, we completely missed one of the main features of the graph (which occurs *between* the values of x=0 and x=1).

If we only rely on a table of values, we can never be sure that we didn't miss an important feature of a graph.



3. Use the Euclidean Distance formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

a) The distance is d= $\sqrt{52}$

b) The distance is d= $\frac{\sqrt{41}}{12}$

4.

a) $(x-3)^2+(y+1)^2=16$

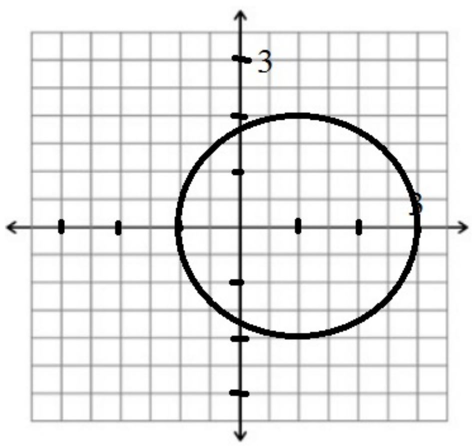
b) $x^2+y^2=1$

c) $(x-1)^2+(y-1)^2=4$

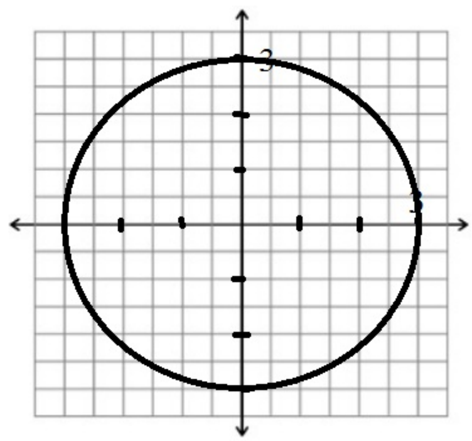
(The easiest method in this case is to sketch the three points on graph paper. You will see that they are all 2 units away from (x,y)=(1,1), hence the centre is (x,y)=(1,1) and the radius is r=2)

5.

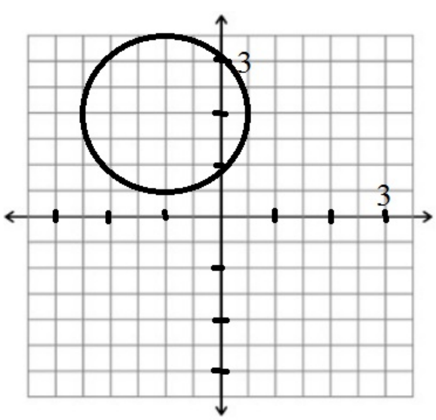
a) Centre is (x,y)=(1,0)
Radius is r=2



b) Re-arrange as x2+y2=9
Centre is (x,y)=(0,0)
Radius is r=3



c) Centre is (x,y)=(-1,2)
Radius is r=1.5



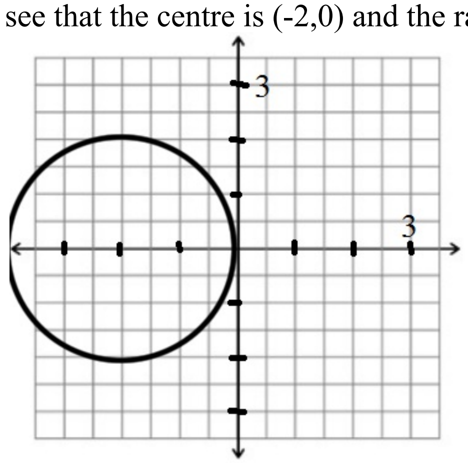
d) Note that this is not yet in the standard form, we have to re-arrange first by completing the square on the x-terms:

$x^2 + 2x + y^2 = 0$

$(x^2+2x+4) - 4 + y^2 =0$

$(x+2)^2 + y^2 = 4$

Now we see that the centre is (-2,0) and the radius is r=2.

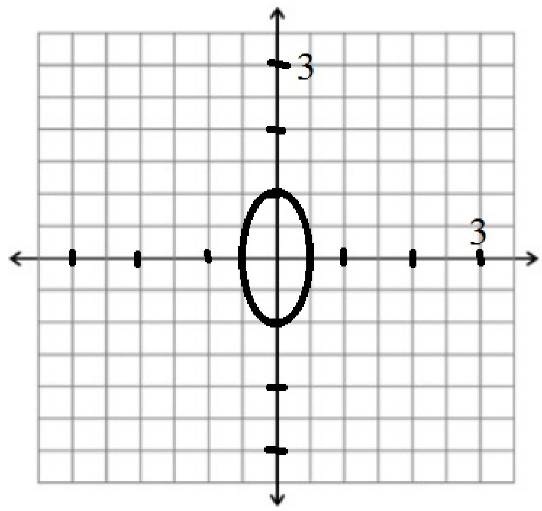


6.

a) Determine the semi-axes by looking at the x- and y- intercepts:

Set x=0 get $y^2 = 1$
so the y-intercepts are $y=\pm 1$

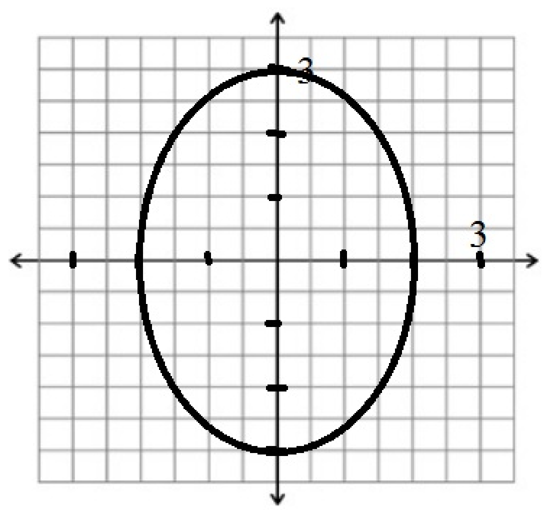
Set x=0 get $4x^2 = 1$
so the x-intercepts are $x=\pm \frac{1}{2}$



b) Determine the semi-axes by looking at the x- and y- intercepts:

Set x=0 get $4y^2 = 36$
so the y-intercepts are $y=\pm 3$

Set x=0 get $9x^2 = 36$
so the x-intercepts are $x=\pm 2$



Unit 8. Linear Equations and Models

1. Find the equation of the line in form $y = mx + b$

    a) with slope 6 passing through point $(-\frac{1}{2}, 2)$

    b) passing through points $(-1,-1)$ and $(1,3)$

    c) passing through points $(4,3)$ and $(-2,5)$

    d) parallel to $y=-3x+2$ that passes through the point $(2,1)$

    e) perpendicular to $4x-3y+6=0$ that passes through the point $(0,-3)$.

2. Sketch the equations of the following three lines:

    a) L1: the line with intercept $b=-3$ and slope $m=+3$

    b) L2: the line with equation $y = \frac{x}{2} - \frac{1}{2}$

    c) L3: the line that passes through point $(x,y)=(7,3)$ with slope $m=-{}^{3}/_{4}$.

    Note that the three lines enclose a triangle. Find the coordinates of the triangle vertices.

3. Find the intercept point between the following pairs of lines:

    a) the lines with equation $y = 3 + 5x$ and $y = 2x - 1$

    b) the line $y = \frac{1}{2} x - \frac{1}{2}$ and the horizontal line at height $y=3$

    c) the lines with equation $3x + 2y = 6$ and $x - y = 3$

4. Setup and solve an inequality to determine the range of x-values for which the graph of $y = 3x + 5$ lies above the graph of $y = 2 - x$

5. If you can sell 20,000 items at a price of \$0.50 each, and 30,000 items at a price of \$0.40 each, find the demand function (i.e. and equation that models price in terms of quantity), assuming it is linear.
    At what price can you sell 32,000 items?
    How many items can you sell at \$0.27 each?

6. A bookseller is trying to liquidate a large warehouse filled with used Calculus texts. A market survey indicates that he could sell 5,000 textbooks each semester at a price of $80 each. Raising the price to $100 each would drop sales to 4,000 textbooks per semester. Find an equation for the price of the textbook in terms of the quantity sold, assuming it is linear. What do the x- and p-intercepts signify?

7. a) A smart-phone plan has a $20/month charge, with no "free" minutes. Each minute of airtime is charged 5 cents. Find the monthly cost C of this phone in terms of the number of airtime minutes x.
   b) A second plan option has no monthly charge, but each airtime minute costs 7 cents. How should you decided between the plan options in (a) and (b)?

8. Let t be the time (in hours) after the start of the experiment. Let T be the temperature (in degrees Centigrade). If after 3 hours the temperature is 52º C, and the temperature drops at a steady rate of 2ºC every 20 minutes,

   a) express T as an equation in terms of t.

   b) A second experiment starts at freezing temperature T=0ºC. and increases steadily at ten degrees per hour. At what time will the two environments have equal temperature?

Unit 8 - Solutions

1.

a) The equation is $y=6x+5$

b) Find the slope using m=rise/run: $m = (3+1) / (1+1)$
$= 2$

Now find the intercept by substituting one point and the slope into $y=mx+b$,
e.g. use the point (-1,-1): $(-1) = (2)(-1) + b$
Solve for $b=1$

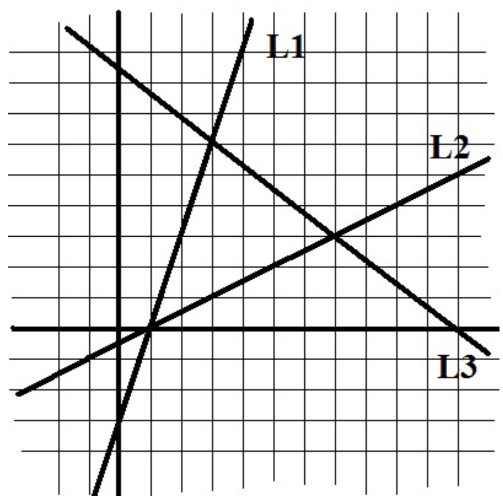
The equation is $y=2x+1$

c) Similar to (b), the final equation is $y=-1/3\ x + 13/3$

d) $y=-3x+7$ (Note: parallel lines have equal slopes, so $m=-3$ is given, then find the intercept as in (b))

e) $y=-3/4\ x - 3$ (Note: the slopes of perpendicular lines are negative reciprocals, i.e our slope must be $m=-3/4$)

2.



The vertices are:
(1,0), (3,6), and (7,3)

Note: in this case we can read them off the graph. We can also find them by setting pairs of equations equal to each other.

E.g. L1 and L2 meet at
$3x - 3 = x/2 - 1/2$
Solve for $x=1$, $y=0$.

3.

a) Set the equations of both lines equal to each other to solve for x, then substitute to find y.

Solve $3 + 5x = 2x - 1$
$3x = -4$
$x = -4/3$. To find y, substitute x into either line equation:
e.g. $y = 3 + 5(-4/3) = -11/3$

The intercept is $(x,y) = (-4/3, -11/3)$

b) Note the the second line simply has equation $y=3$.

Hence solve $\frac{1}{2} x - \frac{1}{2} = 3$ for the intercept $(x,y) = (7, 3)$

c) First state both equations in form $y=mx+b$:

$y = 3 - 3/2\ x$ and $y = x - 3$

Now proceed as in (a) and solve for the intercept $(x,y) = (12/5 , -3/5)$

4. Solve the inequality

$$3x + 5 > 2 - x$$
$$4x > - 3$$
$$x > -3/4$$

The condition is satisfied for x-values in range $x>-3/4$

5. Since the demand/price equation p is linear, we have $p = mx + b$. To find m and b use two points $(x,p) = (20000, 0.5)$ and $(x,p) = (30000, 0.4)$.

We get $m = - .1 / 10000$ and $b = .7$, hence $p = - x/100{,}000 + .7$

$= - 1/ 100{,}000$

Now use this equation to find:

In order to sell x = 32,000 items, you need a price of p=\$0.38.

If you charge p=\$0.27, you can sell x=43,000 items.

6.

We have a linear equation, i.e. $p = mx + b$

Here $m = \text{rise/run} = - 20/1000 = -1/50$

Intercept: Insert a point into $p=mx+b$

e.g. $100 = (-1/50)(4000) + b$, solve for $b=180$

The price equation is $p = -1/50\ x + 180$

The y-intercept ($y=180$) is the maximum price the bookseller "could" charge. In fact, charging \$180 per text would drop his sales to zero ($x=0$).

The x-intercept ($x=9000$) is the maximum quantity, i.e. the number of books the bookseller could get rid of per semester if he was giving away the books for free ($p=0$).

7.

a) This is a linear equation with intercept 20 and slope 0.05,
hence $C(x) = 0.05x + 20$

b) The first plan only makes sense if a lot of monthly airtime is used.

To find the exact cut-off time, setup an inequality to see when first plan is cheaper than the 2nd,
i.e. $0.05\,x + 20 < 0.07\,x$
Solve for $x > 1000$.

I.e. only if you use more than 1000 monthly minutes does the first plan turn out to be cheaper. Otherwise, the 2nd plan is the better investment.

8.

a) Note that this is a linear model. The slope is given by m=rise/run - 2 / (1/3) = -6
(note: 20 minutes is 1/3 hour)

It passes through point $(t,T) = (3,52)$
solve $T = mt + b$ for the intercept: $52 = (-6)(3) + b$, i.e. $b=70$

The equation is $T = -6t + 70$

b) The equation for this environment is $T = 0 + 10t$

Equate (a) and (b) to get $-6t + 70 = 10\,t$.
Solve for time $t=70/16$ hours
or $t=4.375$ hours (or 4 hours and 22 ½ minutes).

<u>Unit 9.</u> <u>Quadratic Equations and Models</u>

1. Find the x-intercepts (if any) of the parabola
   a) $y = 1-x^2$
   b) $y = 2x^2 - 4x - 30$
   c) $y = x^2-4x+4$
   d) $y = x^2+1$

2. In each case, complete the square to write the parabola in the form $y=A(x-B)^2+C$.
   Then describe the shape of the parabola in your own words (vertex, direction, etc).
   Finally, sketch a graph with the given information.
   a) $y = x^2 + 2x + 4$ b) $y = -4x^2 + 12x - 9$ c) $y = -2x^2 -8x + 3$

3. In each case, find the point(s) (x,y) at which the two graphs intersect.
   a) $y= x+1$ and $y = x^2 + 1$
   b) $y= x^2$ and $y = 3 - x^2$

4. A toy rocket is launched vertically into the air from the top of a building. The rocket rises to a maximum height, then begins to fall, missing the building on its way down, and eventually shattering on the ground below. Its height above ground level (in feet) in terms of time (in seconds) is given by the equation $h=-16t^2 + 96t + 256$.
   a) What is the height of the building?
   b) At what time does the rocket hit the ground?
   c) What is the maximum height attained by the rocket?

5. During its initial boosting stage, a rocket is kept steady at a velocity of 10m/sec. After 10sec, the boosters cut out, and gravity pulls the rocket back to earth, its height for the remainder of the voyage given by $y=-5t^2+110t-500$, where t is the time (in seconds) since launch.
   a) At what height will the boosters cut out?
   b) At what time will the rocket hit the ground?
   c) Find the highest point of the rocket's flight path.

6. ABC Electronics is marketing a new TV. Total revenue *R* when x units are sold daily is given by the equation $R = 280x - .4x^2$
   The total cost of producing *x* TVs each day is given by $C=5000 + .6x^2$
   a) What is the daily profit P as an equation in terms of x?
   b) What is the total profit when 100 TVs are produced and sold?
   c) What is the optimal production size that will give you maximum possible profit?
   d) What is the selling price of each TV that will give you maximum possible profit?

7. Solve the quadratic inequality $x^2 + 6x - 7 < 0$

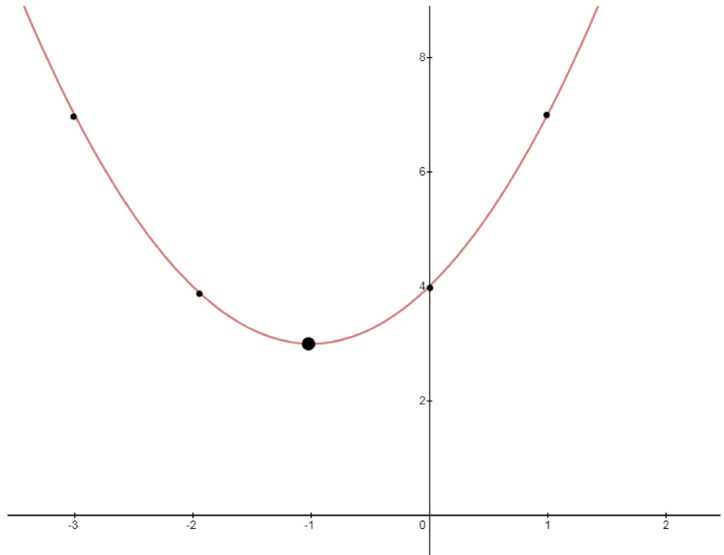
Unit 9 - Solutions

1.
To find x-intercepts, set y=0 and solve for x (by factoring or using the quadratic formula).
a) x=1 and x=-1
b) x=5 and x=-3
c) x=2 (only one x-intercept)
d) the equation $0=x^2+1$ has no solution, hence there are no x-intercepts. This parabola lies entirely above the y-axis.

2.
a) Complete the square

$$y=(x^2+2x+1)+3$$
$$= (x+1)^2+3.$$

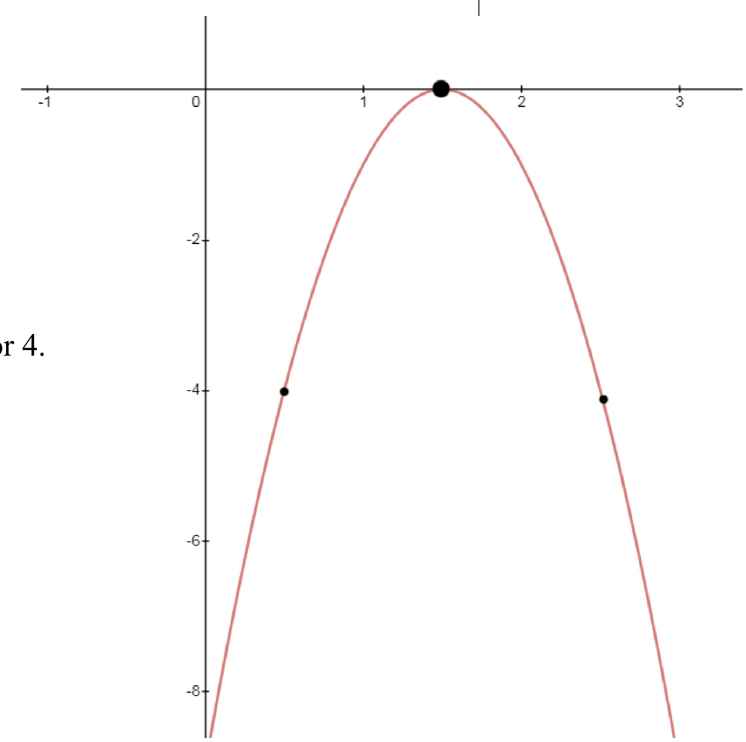
This parabola has vertex (-1,3), opens upward.



b) Complete the square

$$y=-4(x^2-3x)-9$$
$$= -4(x^2-3x)-9$$
$$= -4(x-3/2)^2$$

This parabola has vertex (3/2,0), opens downward, and is stretched vertically by factor 4.

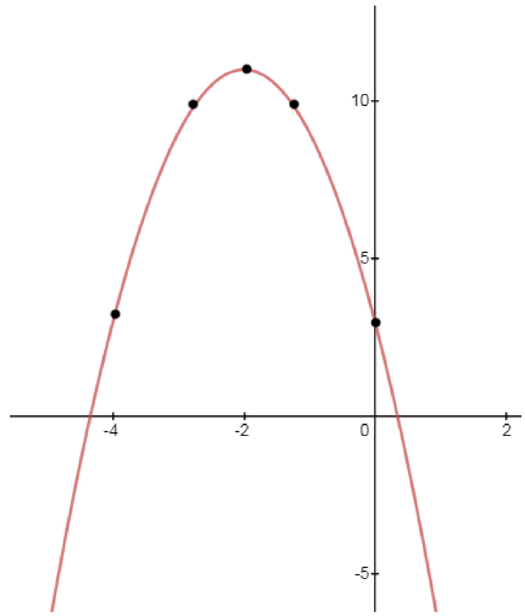


c) Complete the square $y=-2(x^2+4x)+3$

$= -2(x^2+4x+4)+11$

$=-2(x+2)^2 +11$

This parabola has vertex (-2,11),
opens downward,
and is stretched vertically by factor 2.



3.

a) Solve $x+1=x^2+1$ for x=0 or x=1,

i.e. two intersection points (x,y)=(0,1) and (x,y)=(1,2).

b) Solve $x^2=3-x^2$ for $x=\pm\sqrt{(3/2)}$

i.e. two intersection points $(x,y)=(\sqrt{(3/2)},3/2)$ and $(x,y)=(-\sqrt{(3/2)},3/2)$

4.

a) Height of building is the h-intercept (when time t=0), i.e. h=256 feet.

b) Time at which the rocket hits the ground is the t-intercept, i.e. set height h=0 and solve $0=-16t^2 + 96t + 256$.

We get two solutions, t=-2 (discard) and t=8. The rocket hits the ground after 8 seconds.

c) The maximum height is the vertex. Find it by completing the square (or alternate method):

$h = -16(t^2-6t) + 256$

$= -16 (t^2-6t+9-9) + 256$

$= -16 (t^2-6t+9) +144 +256$

$= -16 (t-3)_2 + 400$

The maximum height occurs after 3 seconds and is 400 feet.

5.

a) The boosters cut out after t=10 seconds. The rocket will have reached a height of 100 metres.

b) We only consider the second part of the rocket's trip, and set height y=0,
to solve $0=-5t^2+110t-500$

For $t=(-110\pm\sqrt{2100})/(-10)$ so $t_1 \approx 6.41$ sec (doesn't count, less than 10 sec)
$t_2 \approx 15.582$ sec.
Hence the rocket hits the ground after about 15.6 seconds.

c) Again, we only consider the second part of the rocket's path.

Complete the square to get $y=-5(t-11)^2 + 105$

The maximum height is 105 metres.

6.
a) Profit = Revenue - Costs,
hence $P = 280x - .4x^2 - (5000 + .6x^2)$
$= -x^2 +280 x - 5000$

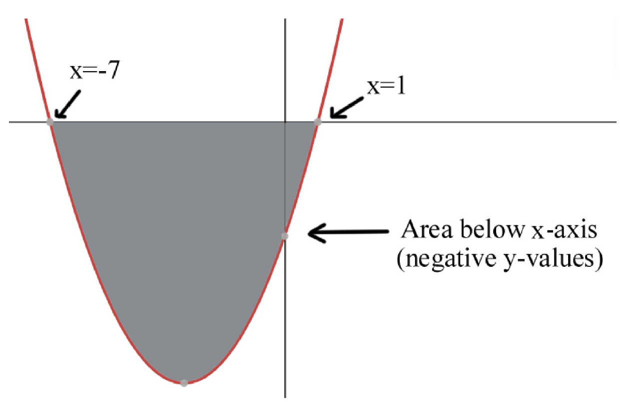
b) Substitute x=100 to find P=13000. Hence the profit is $13,000.
c) Find the vertex of the profit equation(either by completing the square or finding the midpoint between the two x-intercepts) at x=140. Hence the optimal daily production size is 140 TVs.
d) When x=140 TVs are sold, the total revenue is R=31360. Hence the prive per TV is $224.

7.

Note that the equality has solutions $(x-1)(x+7)=0$, i.e. x=1 and x=-7.

Since the graph of the quadratic is a parabola opening upward, and we are looking for x-values that produce negative y-values, our solution must be all values between the intercepts,

i.e. the solution is all x-values strictly between $-7<x<1$



Unit 10. Functions: Basic Properties

1. In each case, find the domain of the given function:

a) $f(x) = 3x^2 + x - 1$

b) $z(x) = \dfrac{3x}{x-1}$

c) $f(x) = \dfrac{x+1}{x^3 + x^2 - 6x}$

d) $g(z) = \sqrt{5z - 6}$

e) $f(x) = x + \sqrt{x} - \dfrac{1}{x}$

f) $h(t) = \sqrt{3 - t^2}$

g) $f(x) = \dfrac{x}{\sqrt{3 - x^2}}$

h) $f(x) = \left(x^2 - 2x - 5\right)^{1/3}$

i) $g(t) = \sqrt{t^2 - 3t + 2}$

2. In each case, find the range of the given function:

a) $f(x) = x^2 + 1$

b) $f(x) = 3 - 5x$

c) $f(x) = 3 - 2x - x^2$

d) $f(x) = \dfrac{1}{4 + x^2}$

3. Given $f(x) = x^2 + 2x$, find and simplify an expression for

a) $f(-3)$

b) $f(x-1)$

c) $f(x+1) - 3f(x)$

d) $x^2 - xf(2x)$

e) $\dfrac{f(x+1) - f(1)}{x}$

4. Repeat Question #3 a-e using the function $f(x) = \dfrac{x}{2 - x}$

Unit 10 - Solutions

1. Remember in all of these questions that the domain is the set of x-values that you are allowed to use in a given function. It is often easier to turn the question around, and to look for x-values that you are *not* allowed to use instead.

a) There are no restrictions on x. The domain is all $x \in R$.

b) We cannot divide by zero, hence we must exclude all x for which $x - 1 = 0$,
i.e. the domain is all $x \in R$ except $x=1$.

c) Here we must ensure that $x^3+x^2-6x \neq 0$.
Where is $x^3+x^2-6x=0$? Factor to find that
$x(x-2)(x+3)=0$ if $x=0$, $x=2$, or $x=-3$.
Hence your domain is *any* x-value *except* $x=0$, $x=2$, $x=-3$.

d) You are not allowed to take a square root (or any even root) of a negative number, hence we need $5z - 6 \geq 0$
Hence the domain is $z \geq 6/5$, or using interval notation $z \in [6/5,\infty)$.

e) We have two restrictions, x cannot equal zero (because of the division) and x cannot be negative (because of the root). Put them together, and the domain is all $x>0$.

f) Because of the root, we need $3 - t^2 \geq 0$.
Factor to get $(\sqrt{3} - t)(\sqrt{3} + t) \geq 0$.
Since $y=3-t^2$ is the graph of a downwards opening parabola with intercepts at $\pm\sqrt{3}$, it will be greater than or equal to zero when $-\sqrt{3} \leq t \leq \sqrt{3}$.
Hence the domain is $-\sqrt{3} \leq t \leq \sqrt{3}$ or in interval notation $t \in [-\sqrt{3},\sqrt{3}]$.

g) This is very similar to (f), except that we are also dividing by the expression, hence we now need the strict inequality $3-x^2>0$.
The domain for this function is hence $-\sqrt{3} < x < \sqrt{3}$ or alternatively $x \in (-\sqrt{3},\sqrt{3})$.

h) We are taking a third root, which has no restrictions (for example, the third root of -8 is -2, since $(-2)(-2)(-2)=(-8)$). Hence there are no domain restrictions, the domain is all real numbers, or $x \in R$.

i) Proceed similar to f) above. Here $t^2-3t+2=(t-1)(t-2)$, hence the expression under the root is zero if $t=1$ or $t=2$. Since $y=t^2-3t+2$ is an upward opening parabola, the domain of g(t) is all $t \leq 1$ and $t \geq 2$.

2.

a) This is an upward opening parabola with vertex (x,y)=(0,1),
hence the range is all $y \geq 1$.

b) This is a straight line... every y-value will be reached for some x-value,
hence the range is all $y \in R$.

c) This is a downward opening parabola with vertex (x,y)=(-1,4),
hence the range is all $y \leq 4$.

d) Note that $y=4+x^2$ will have a range of $y \geq 4$,
hence the reciprocal will have a range of $y \leq 1/4$

3.

a) $f(-3) = 9 - 6 = 3$

b) $f(x-1) = (x-1)^2 + 2(x-1) = x^2 - 1$

c) $f(x+1) - 3f(x) = (x+1)^2 + 2(x+1) - 3(x^2 + 2x)$

$= -2x^2 - 2x + 3$

d) $x^2 - xf(2x) = x^2 - x((2x)^2 + 2(2x)) = -4x^3 - 3x^2$

e) $$\frac{f(x+1) - f(1)}{x} = \frac{(x+1)^2 + 2(x+1) - (1^2 - 2)}{x}$$

$$= \frac{x^2 + 4x}{x} = x + 4 \qquad \text{(if } x \neq 0\text{)}$$

4.

a) $f(-3) = -3/5$

b) $$f(x-1) = \frac{x-1}{3-x}$$

c) $$f(x+1) - 3f(x) = \frac{x+1}{1-x} - 3\frac{x}{2-x} = \frac{2x^2 - 2x + 2}{(1-x)(2-x)}$$

d) $$x^2 - xf(2x) = x^2 - x\frac{2x}{2-2x} = \frac{-x^3}{1-x}$$

e) $$\frac{f(x+1) - f(1)}{x} = \frac{\frac{x+1}{2-(x+1)} - \frac{1}{1}}{x} = \frac{\frac{2x}{1-x}}{x} = \frac{2}{1-x} \qquad \text{(if } x \neq 0\text{)}$$

Unit 11. Graphs of Functions

1. Which of the following equations define y as a function of x?

   a) $x + 3y = 4$ b) $x^2 + 4y^3 - x = 0$ c) $x^2 - 2y^2 = x$

2. Describe the graph of each of the following equations. Based on the graph, will this equation define y as a function of x?

   a) $y = 3x - 4$ b) $(x-1)^2 + (y+2)^2 = 4$ c) $y = 4x^2 - 3x + 1$

3. Find all x- and y-intercepts of the graphs of both $f(x) = -x^2 - 12x$ and $g(x) = x^2 - x + 12$, as well as any points (state both x and y coordinates) where they intersect each other.

4. Find all x- and y-intercepts of the graphs of both $f(x) = \dfrac{x - x^2}{x}$ and $g(x) = \sqrt{x^2 + 4}$, as well as any points where they intersect each other.

5. Sketch a graph of each of the following split-definition functions:

   a) $f(x) = \begin{cases} 2x - 1 & \text{if} \quad x > 1 \\ 2 - x & \text{if} \quad x \le 1 \end{cases}$ b) $g(x) = \begin{cases} x + 6 & \text{if} \quad x < -2 \\ x^2 & \text{if} \quad -2 \le x \le 1 \\ 2x - 1 & \text{if} \quad x > 1 \end{cases}$

   c) $h(x) = \begin{cases} x^2 + 2x & \text{if} \quad x \le 0 \\ 2x - x^2 & \text{if} \quad x \ge 0 \end{cases}$

6. Using the graphs (a) and (b) in Question #5 above, find for each

   $y = f(x)$ and, $y = g(x)$

   i) all x- and y-intercepts
   ii) all intervals of increase and all intervals of decrease
   iii) all local extreme points

Unit 11 - Solutions

1.

a) Solve for $y=\dfrac{4-x}{3}$ This defines y as a function of x (each x produces only one y)

b) Solve for $y=\sqrt[3]{\dfrac{x-x^2}{4}}$ This defines y as a function of x (there is a unique cube root for every value).

c) Solve for $y^2=\dfrac{x^2-x}{2}$, i.e. $y=\pm\sqrt{\dfrac{x^2-x}{2}}$

This is NOT a function, as each x produces two y-values.

2.

a) This is the graph of a straight line. It satisfies the vertical line test. The equation is a function.

b) This is the graph of a circle (with centre (1,-2) and radius 2). This graph fails the vertical line test, and the equation is NOT a function.

c) This is the graph of a parabola. It satisfies the vertical line test. The equation is a function.

3.

For $f(x)$: y-intercept f(0)=0

x-intercept: solve f(x)=0. Factor -x(x+12)=0 to get solutions x=0 and x=-12.

For $g(x)$: y-intercept is $g(0)=12$

x-intercept: solve $0=x^2-x+12$. This has no solution (check quadratic formula), so g(x) does not have any x-intercepts.

Where does f(x) intersect g(x)? Set equal and solve:

$$-x^2-12x=x^2-x+12$$

$$0=2x^2+11x+12$$

Factor 0=(2x+3)(x+4), hence they intersect at x=-3/2 and x=-4.

To find y-coordinates, plug x-values into either f(x) or g(x).

The two interception points are (x,y)=(-3/2,63/4) and (x,y)=(-4,32).

4. Careful! The domain of $f(x)$ does not include x=0!

For $f(x)$ y-intercept does not exist since $f(0)$ is undefined.

x-intercept: solve $0 = \dfrac{x - x^2}{x}$ for $x = 1$.

(remember x=0 is not part of the domain, so there is only one x-int)

For $g(x)$ y-intercept is $g(0) = 2$

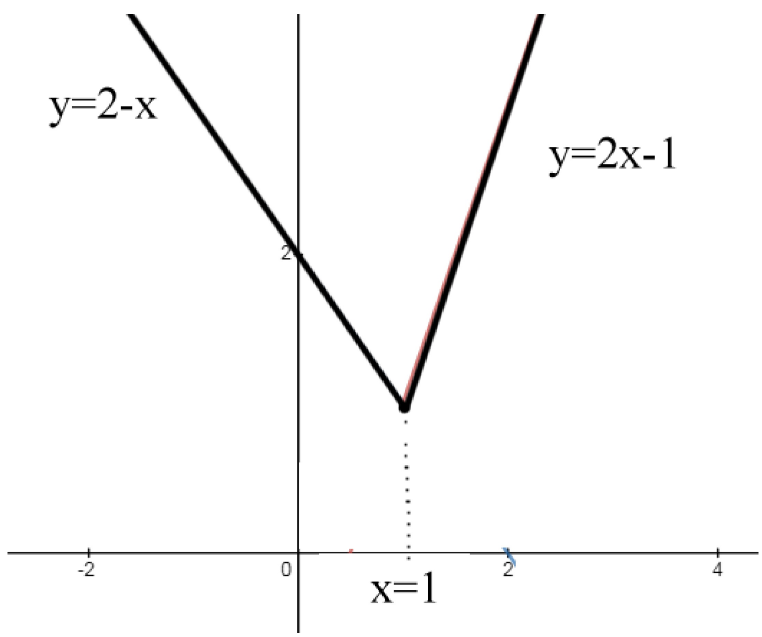
x-intercept: solve $0 = \sqrt{x^2 + 4}$. However, this quadratic has no solution, hence g(x) has no x-intercepts.

Where does f(x) intersect g(x)? Set equal and solve:

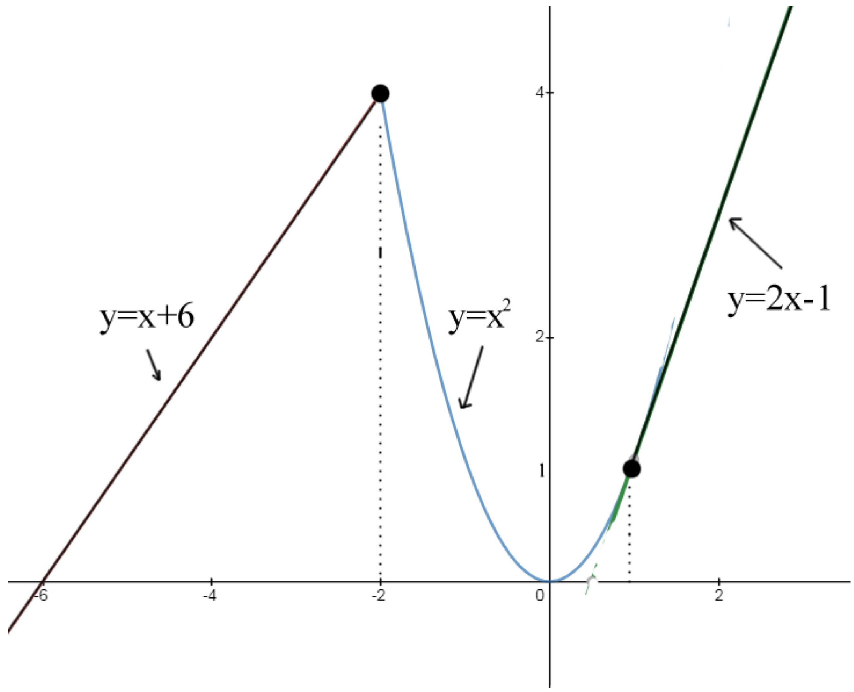
$$\frac{x - x^2}{x} = \sqrt{x^2 + 4}$$
$$x - x^2 = x\sqrt{x^2 + 4}$$
$$x^2 - 2x^3 + x^4 = x^2(x^2 + 4)$$
$$-x^2(2x + 3) = 0$$

This has solutions x=0 (but again not part of the domain of f(x)!) and x=-3/2.
Hence their only intersection point is the point (x,y)=(-3/2, 5/2)

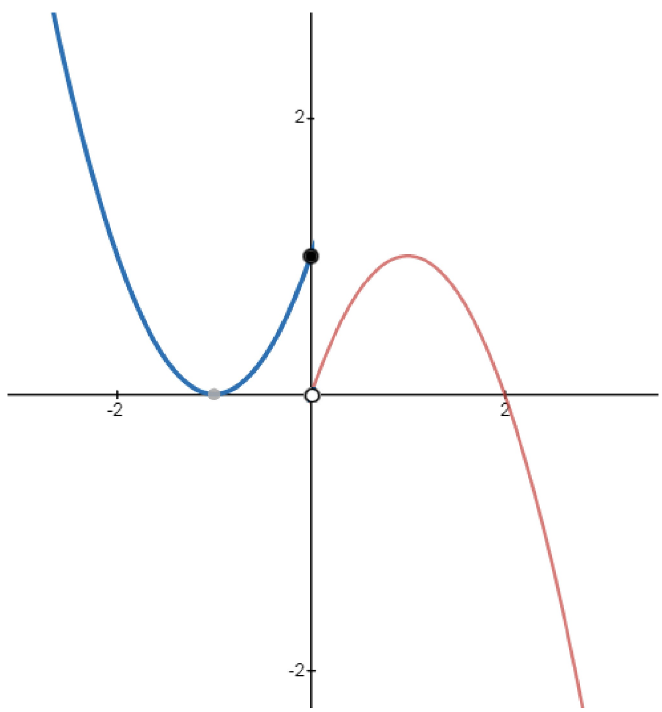
5. a) Each piece is linear. Note that the lines meet up at (x,y)=(1,1).



b) A section of a parabola in the middle, flanked by line pieces on each side. Each piece meets up perfectly at (x,y)=(-2,4) and (x,y)=(1,1).



c) Each section is a parabola piece. However, they do not meet up at the split. Pay careful attention to the inequality signs - the left half has a filled circle, the right half has an open circle.



6. a) i) y-intercept $y=2$, no x-intercepts.
ii) increasing for $x>1$, decreasing for $x<1$
iii) local minimum at $(x,y)=(1,1)$

b) i) y-intercept $y=0$, x-intercept $x=0$ and $x=-6$ (extend graph to left)
ii) increasing for $x<-2$ and $x>0$, decreasing for $-2<x<0$
iii) local maximum at $(x,y)=(-2,4)$, local minimum at $(x,y)=(0,0)$

Unit 12. Transformation of Functions

1. Are the following functions even, odd, both or neither?

a) $f(x) = x^2 + 3x^4 + 1$ b) $f(x) = \dfrac{1+x^3}{x^2+1}$ c) $f(x) = \dfrac{x^3 - 2x}{x^4 + 2}$

2. Given the following functions:

$f(x) = 2x^2 + x - 1$ $g(x) = \dfrac{1}{x+1}$ $h(x) = \sqrt{x+1}$

find and simplify:

a) $f(x)g(x)$ b) $f(g(x))$ c) $g(f(x))$ d) $h(f(x))$

3. Starting with the relevant basic graph, use shifting/scaling to obtain the graph of the given function.

a) $f(x) = (x+2)^3 - 3$ b) $f(x) = 1 - \dfrac{1}{x-1}$

c) $f(x) = \sqrt{1-x}$ d) $f(x) = |2x| - 1$

4. Consider the given graph *y=f(x)*. For each of the given functions below, write a short sentence describing how you would find their graphs using the graph of *y=f(x)*, then sketch each graph on a separate coordinate system.

a) $y = f(x-1) + 2$

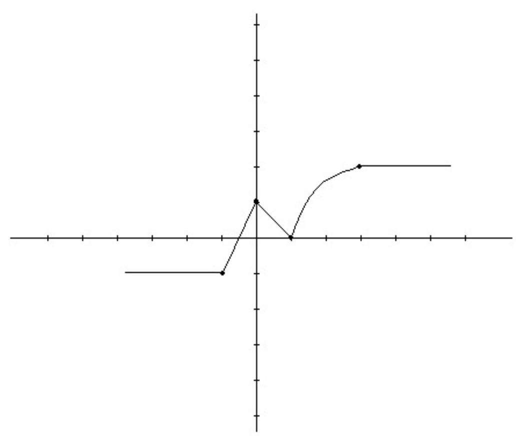
b) $y = f(x+1) - 3$

c) $y = 2\,f(x) + 1$

d) $y = 2 - f(x)$

e) $y = \frac{3}{2} f(-x) - 1$

f) $y = \frac{1}{2} f\left(\frac{2x}{3}\right) + \frac{1}{2}$



5. A spherical balloon is being inflated. At time t=0, its radius is 25 cm, and the radius is increasing at a constant rate of 3 cm per minute.

a) Find the function V(r), i.e. the volume of the balloon as a function of radius.
b) Find the function r(t), i.e. the radius of the balloon as a function of time.
c) Use function composition to find the volume as a function of time.
d) Use (c) to find the time at which the balloon's volume is a half million cm^3.

Unit 12 - Solutions

1. To check the symmetry of a function, evaluate it at “-x”. If f(-x)=f(x), the function is even. If f(-x)=-f(x), the function is odd.

a) $f(-x)=(-x)^2+3(-x)^4+1=x^2+3x^4+1=f(x)$ This is even.

b) $f(-x)=\dfrac{1+(-x)^3}{(-x)^2+1}=\dfrac{1-x^3}{x^2+1}$ This is neither odd nor even.

c) $f(-x)=\dfrac{(-x)^3-2(-x)}{(-x)^4+2}=\dfrac{-x^3+2x}{x^4+2}=-\dfrac{x^3-2x}{x^4+2}$ This is odd.

2.

a) $f(x)g(x)=(2x^2+x-1)(\dfrac{1}{x+1})=\dfrac{2x^2+x-1}{x+1}=\dfrac{(x+1)(2x-1)}{x+1}=2x-1$

(if $x\neq-1$)

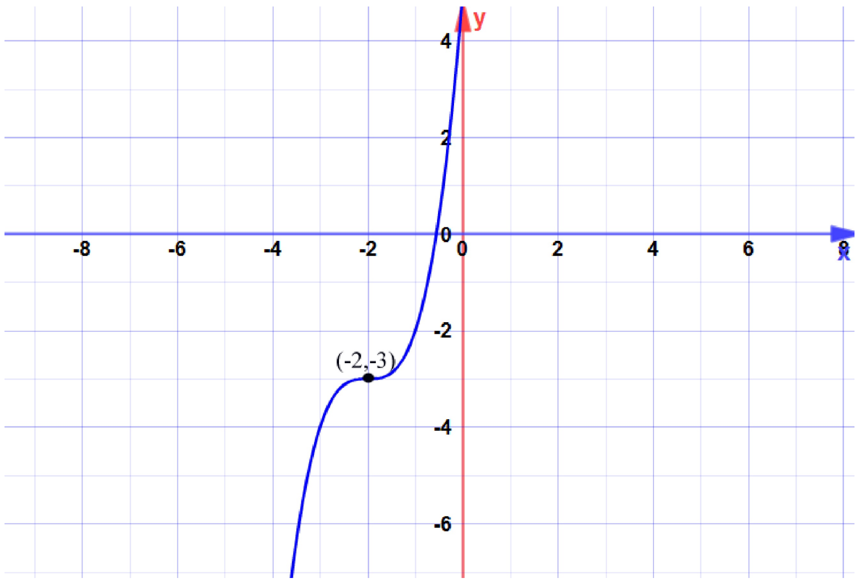
b) $f(g(x))=2(\dfrac{1}{x+1})^2+(\dfrac{1}{x+1})-1=\dfrac{-x^2-x+2}{(x+1)^2}$

c) $g(f(x))=\dfrac{1}{(2x^2+x-1)+1}=\dfrac{1}{x(2x+1)}$

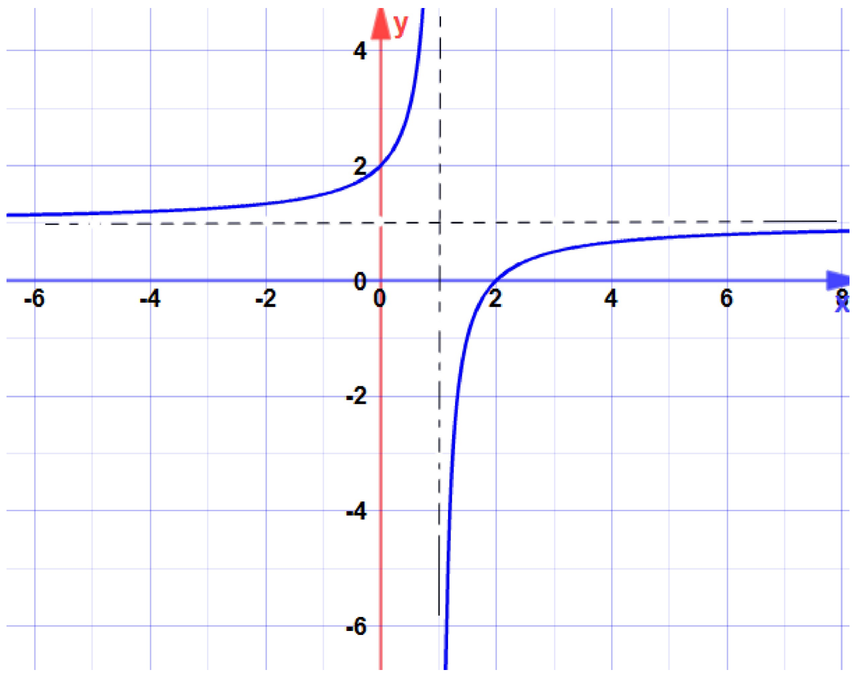
d) $h(f(x))=\sqrt{(2x^2+x-1)+1}=\sqrt{x(2x+1)}$

3.

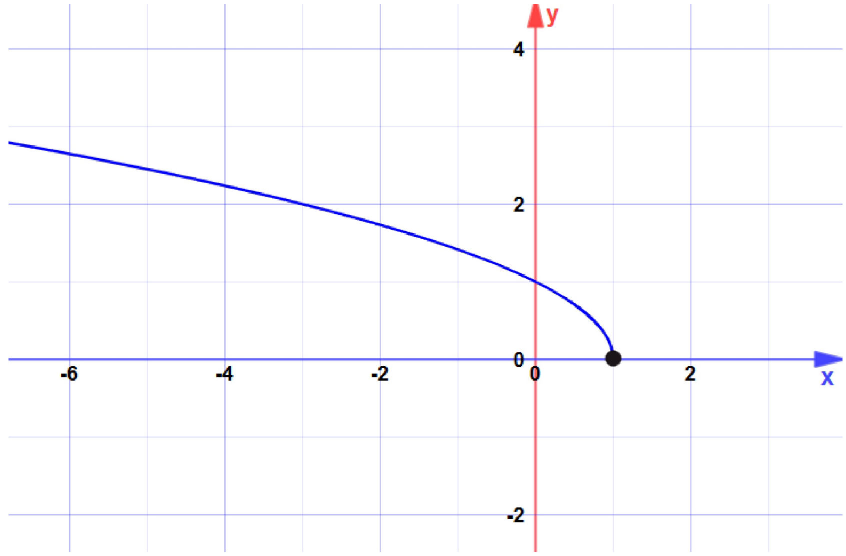
a) Start with base-graph $y=x^3$. (Cubic)
Shift it two units to the left and three units down.



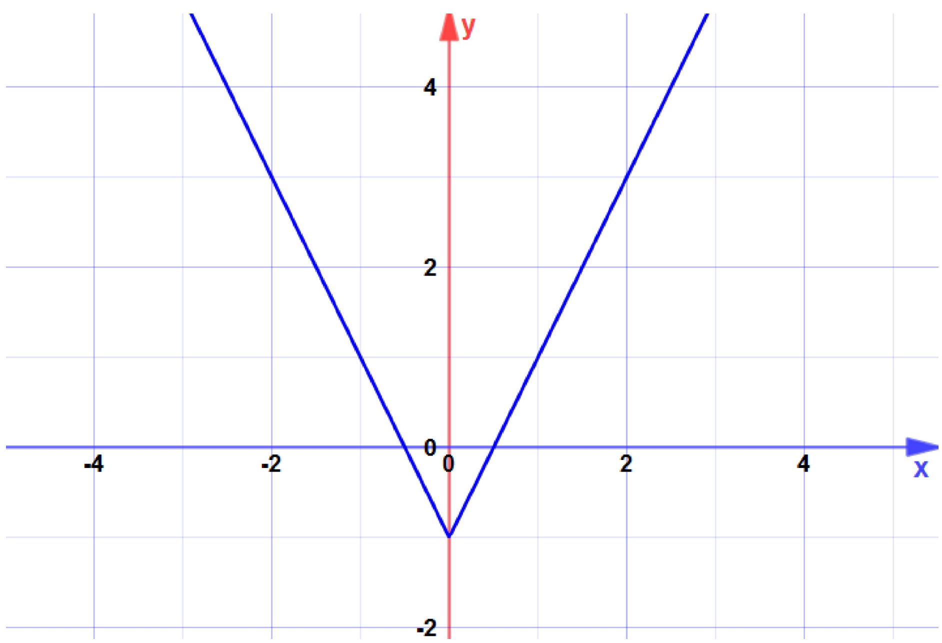
b) Start with base-graph $y=1/x$ (hyperbola)
Flip it vertically ($y=-1/x$).
Shift it one unit to the right: $y=-1/(x-1)$
Shift it one unit up.



c) Start with base-graph $y=\sqrt{x}$ (square root)
Flip it horizontally,
shift it one unit to the right.

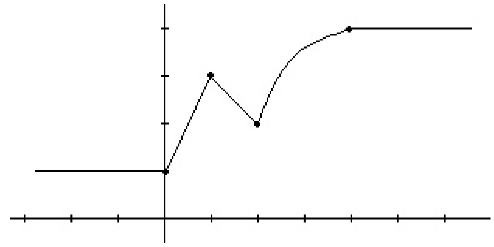


d) Start with base-graph $y=|x|$ (absolute value graph)
Compress it horizontally by factor 2.
Shift down one unit.

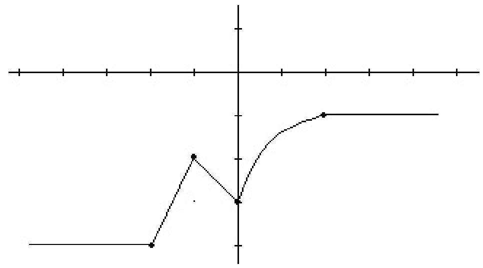


4.

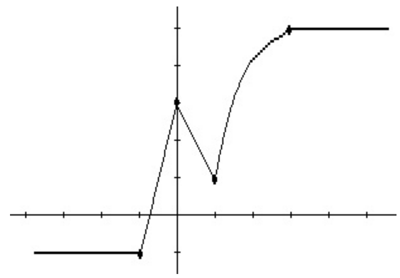
a) $y = f(x-1) + 2$
Shift the graph one unit to the right, and two units up.



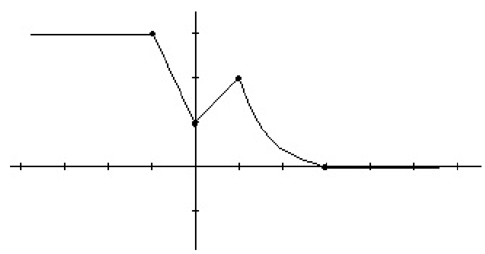
b) $y = f(x+1) - 3$
Shift the graph one unit to the left, and three units down.



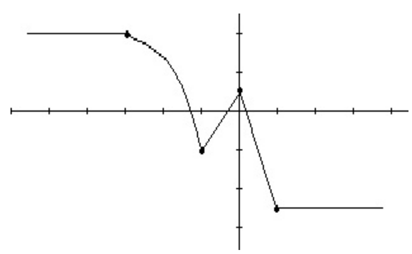
c) $y = 2\ f(x) + 1$
Stretch the graph vertically by a factor of two, then shift it one unit up.



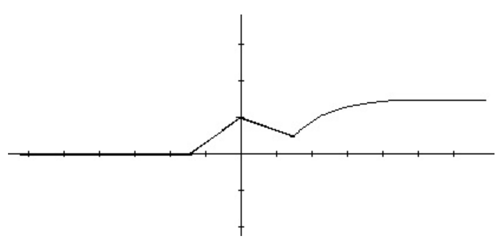
d) $y = 2 - f(x)$
Think of it as $y = -f(x) + 2$.
Flip the graph vertically, then shift it up by two units.



e) $y = {}^{3}/_{2}\ f(-x) - 1$
Flip the graph horizontally, stretch the graph vertically by a factor of 3/2, then shift it down one unit.



f) $y = \frac{1}{2}\ f({}^{2x}/_{3}) + \frac{1}{2}$
Stretch the graph horizontally by a factor 3/2, compress it vertically by a factor 2, then move it up ½ unit.



5.

a) $V(r) = \frac{4}{3}\pi r^3$ (standard geometry formula)

b) $r(t) = 25 + 3t$ (linear equation, similar to Unit 1 problems)

c) $V(t) = \frac{4}{3}\pi(25 + 3t)^3$ (combine/compose the two functions)

d) Set Volume V=500,000 and solve for time t.

$$\frac{1500000}{4\pi} = (25 + 3t)^3$$

i.e. t = $t = \dfrac{\sqrt[3]{\dfrac{1{,}500{,}000}{4\pi}} - 25}{3}$

or roughly t=8 minutes.

Unit 13. Polynomial Functions

1. Consider the polynomial $f(x) = 2x^3 - x^2 + 3$

   Evaluate and simplify:

   a) $f(2)$ b) $f(\frac{2}{3})$ c) $f(\sqrt{2})$

   d) $f(2y+1)$ e) $f(a+1) - f(a-1)$

   f) $xf(x^2) - x^3$ g) $\dfrac{f(x+y) - f(y)}{x}$

2. For each of the following polynomials, determine the degree and the leading coefficient, find and classify all roots, and sketch a graph of the function based on this information.

   a) $f(x) = x^3 - x$ b) $f(x) = -x^5 + 2x^4 - x^3$

   c) $f(x) = (x+1)^3(2-x)^2(2x-1)$ d) $f(x) = -2x^4 + 11x^3 - 15x^2$

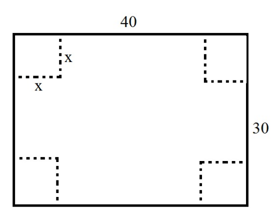
3. For each of the following polynomials, determine the degree and the leading coefficient. In each case, a root is given. Use long division to find and classify the remaining roots. Then sketch a graph of the function based on this information.

   a) $f(x) = 3x^3 + 4x^2 - 17x + 10$ (given root: x=1)

   b) $f(x) = -4x^4 + 12x^3 - 9x^2 + 2x$ (given root: x=2)

   c) $f(x) = -x^5 + 6x^4 + 28x^3 - 110x^2 + 117x - 40$ (given root: x=1 is a triple root)

4. An open box is made from a rectangular piece of cardboard measuring 30cm by 40cm, by cutting away four identical squares from each corner and bending up the sides.



   a) If the cut-away squares each have side length "x", construct the function V(x) for the volume of the resulting box.

   b) Find the volume of the resulting box if the cut-away squares each measure 3cmx3cm. What are the dimensions of the box?

   c) If you need the resulting box to have volume 3000cm^3, how large should the cut-away squares be? (Note: you will need to "guess and check" as the resulting equation does not factor easily)

   d) What is the domain of V(x)? (take into account the physical restrictions of this setup)

Unit 13 - Solutions

1.

a) $f(2) = 15$

b) $f(2/3) = 85/27$

c) $f(\sqrt{2}) = 4\sqrt{2} + 1$

d) $f(2y+1) = 16y^3 + 20y^2 + 8y + 4$

e) $f(a+1) - f(a-1) = 12a^2 - 4a + 4$

f) $xf(x^2) - x^3 = 2x^7 - x^5 - x^3 + 3x$

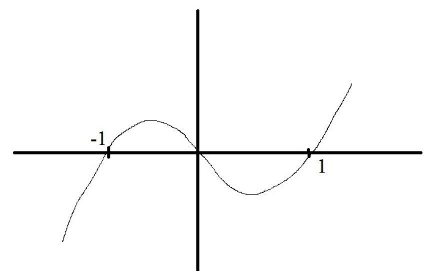
g) $\dfrac{f(x+y) - f(y)}{x} = 2x^2 + 6xy + 6y^2 - x - 2y$ (if $x \neq 0$)

2.

a) Degree 3, Leading Coefficient 1 (positive).

$f(x) = x(x-1)(x+1)$

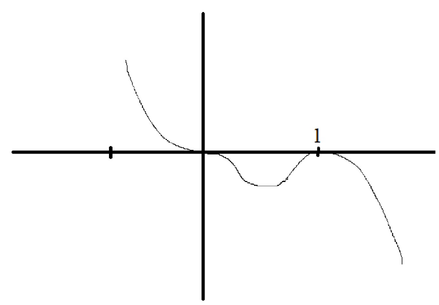
Roots: x=0 (single), x=1 (single), x=-1 (single)



b) Degree 5, Leading Coefficient -1 (negative).

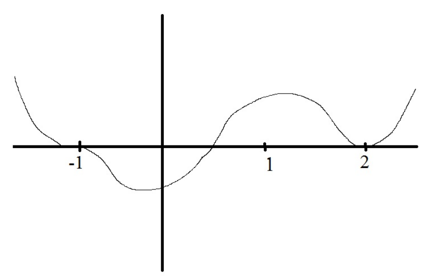
$f(x) = -x^3(x-1)^2$

Roots: x=0 (triple), x=1 (double)



c) Degree 6, Leading Coefficient 2 (positive).

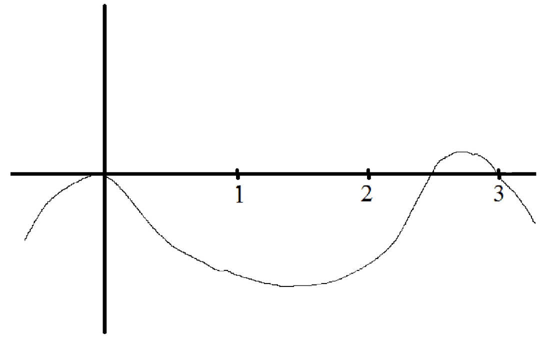
Roots: x=-1 (triple), x=2 (double), x=1/2 (single)



d) Degree 4, Leading Coefficient -2 (negative).

$$f(x) = -x^2(x-3)(2x-5)$$

Roots: x=0 (double), x=3 (single), x=5/2 (single)



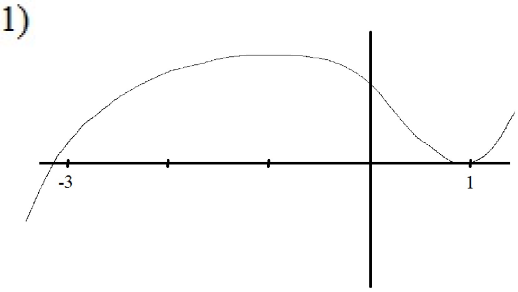
3.

a) Degree 3, Leading Coefficient 3 (positive).

Long division: $(3x^3 + 4x^2 - 17x + 10) \div (x-1) = (3x^2 + 7x - 10)$

Factor: $f(x) = (x-1)(3x+10)(x-1)$

Roots: x=1 (double), x=-10/3 (single)

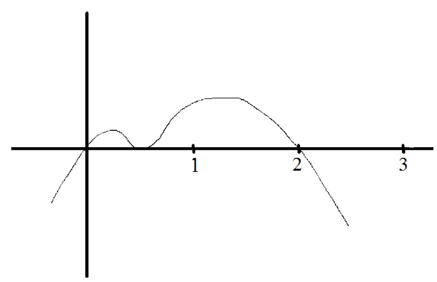


b) Degree 4, Leading Coefficient -4 (negative).

Long division: $(-4x^4 + 12x^3 - 9x^2 + 2x) \div (x-2) = -4x^3 + 4x^2 - x$

Factor: $f(x) = -x(x-2)(2x-1)^2$

Roots: x=0, x=2, x=1/2 (double)



c) Degree 5, Leading Coefficient -1 (negative).

Long division (three times!):

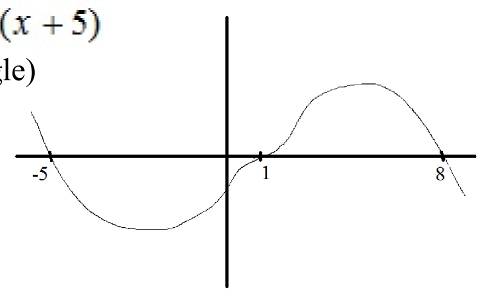
First: $(-x^5 + 6x^4 + 28x^3 - 110x^2 + 117x - 40) \div (x-1) = -x^4 + 5x^3 + 33x^2 - 77x + 40$

Next: $(-x^4 + 5x^3 + 33x^2 - 77x + 40) \div (x-1) = -x^3 + 4x^2 + 37x - 40$

Next: $(-x^3 + 4x^2 + 37x - 40) \div (x-1) = -x^2 + 3x + 40$

Finally we can factor: $f(x) = -(x-1)^3(x-8)(x+5)$

Roots: x=1 (triple), x=8 (single), x=5 (single)



4.

a) The resulting box will have length 40-2x, width 30-2x, height x.

Hence volume $V(x) = (40-2x)(30-2x)(x) = 4x^3 - 140x^2 + 1200x$

b) Set x=3: V(3)=2448 cm^2

c) Set V=3000: Solve $3000 = 4x^3 - 140x^2 + 1200x$ for x=5.

Hence we need to cut away 5cm x 5cm squares. The dimensions of the resulting box would be 30 cm by 20 cm by 5 cm.

(Note: there are two other roots, $x = 15 \pm 5\sqrt{3}$. See comment in d))

d) The polynomial itself has no restrictions, but we need to consider the physical restrictions of the problem.

Clearly x>0 is necessary (cannot cut away a negative square).

Further x<15 is required (cannot cut away more from the 30cm side)

So the domain is 0<x<15.

(Note from (c): We now see that $x=15+5\sqrt{3}$ is not in the domain.

However, $x=15-5\sqrt{3} \approx 6.3$ cm is. So another solution to (c) is a box with dimensions approximately 27.4 cm by 17.4 cm by 6.3 cm.)

Unit 14. Exponential and Logarithmic Functions

1. Use the exponent rules to simplify each expression: a) $\dfrac{t^{-2/3}t^{3/4}}{t}$ b) $\sqrt{\dfrac{x^3y^6}{xy^4}}$

2. Write the given expressions as a single exponential, e.g . $\dfrac{2^{3x}}{2^2}$ can be written as a 2^{3x-2}.

a) $(2^{3+x})\dfrac{3^x}{2^3}$ b) $e^{3x}(e^{2x})^2$ c) $\dfrac{e^{5x/3}}{\sqrt[3]{e^x}}$

3. Use the exponent rules to solve for x:

a) $3^{x-1} = 81$ b) $(1/2)^{x+1} = (1/4)^{x+2}$ c) $2^{x^2} = 4^{x-1}$

4. Sketch a graph of the functions $y = (3/2)^x$ and $y = 6 - 3^x$

5. Write the given expression as a single logarithm, if possible.

a) $\ln(2x) + \ln(x) - \ln(3x)$ b) $4\ln(x) - \frac{1}{2}\ln(3x) + 1$

c) $\ln(x)\cdot\ln(2x)$ d) $4\log(x+1) - 2\log(x) + 2$

6. Find the value of \$2000 in ten years if it grows at 7% annually, and interested rate is accumulated a) yearly b) monthly c) daily d) continuously.

7. Solve each expression for x:

a) $2^{3x} = 2\cdot 2^x$ b) $4e^{3x} = 16$ c) $\ln(2\ln(t)) = 0$ d) $e^{\ln(x^3)} = 2x\ln(e^2)$

e) $2 = 10^{4x-2}$ f) $7 = 3e^{0.4t}$ g) $\ln(\ln(x)) = 1$ h) $\ln(2x+1) = 2 - \ln(x)$

8. Sketch a graph of each function (using shifting/scaling):

a) $f(x) = 2^{x-1} - 1$ b) $f(x) = \ln(x+1) + 1$

c) $f(x) = e^{-x} - 1$ d) $f(x) = -\ln(x+2)$

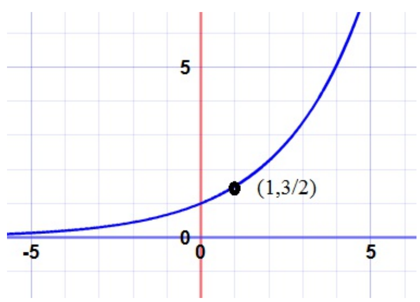
Unit 14 - Solutions

1.
a) $t^{-11/12}$ b) xy (more precisely, |xy|, since the result cannot be negative)

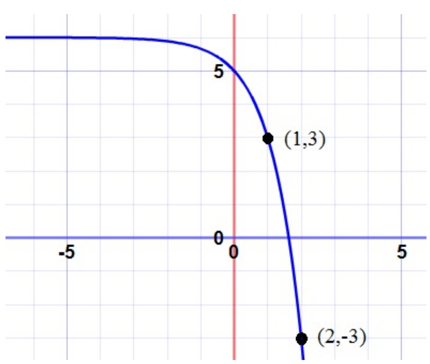
2.
a) $2^{3+x} / 2^3$ can be simplified to 2^x (base is the same, so subtract exponents).
Now $2^x \cdot 3^x$ (different base but same exponent) can be simplified to just final answer 6^x
b) First square e^{2x} to get e^{4x} (multiply exponents), now combine with e^{3x} (add exponents) to get final answer e^{7x}
c) Denominator can be written as $e^{x/3}$. Now subtract exponents to get final answer $e^{4x/3}$.

3.
a) Rewrite as $3^{x-1} = 3^4$, solve x-1=4 for x=5.
b) Rewrite as $(1/2)^{x+1} = (1/2)^{2x+4}$ and solve x+1 = 2x+4 for x=-3
c) Rewrite as $2^{x^2} = 2^{2x-2}$ and solve $x^2 = 2x - 2$
However, the discriminant of this equation is negative, hence there are no solutions.

4. $y = (3/2)^x$



$y = 6 - 3^x$ (the basic 3^x graph flipped vertically and moved up 6 units)



5.

a) Remember that ln(A)+ln(B)=ln(AB) and ln(A)-ln(B)=ln(A/B).

Combine the three logarithms to get ln ((2x)(x)/(3x)) = ln (2x/3)

b) Proceed as in (a), but first rewrite $4\ln(x)=\ln x^4$, $\frac{1}{2}\ln(3x) = \ln (3x)^{1/2}$ and $1=\ln(e)$.

Now combine to get $\ln x^4 - \ln (3x)^{1/2} + \ln(e) = \ln (ex^4 / (3x)^{1/2})$

c) This cannot be further simplified... there is no logarithm rule that allows the simplification of a product of two logarithms.

d) Note that 2=log(100) since $10^2=100$ (you need everything in terms of logs to combine).

$$\log(x+1)^4 - \log x^2 + \log(100) = \log\left(\frac{100(x+1)^4}{x^2}\right)$$

6 .

a) $B = 2000\,(1.07)^{10} = \$3934.30$

b) $B = 2000\,(1 + 0.07/12)^{120} = \$4019.32$

c) $B = 2000\,(1 + 0.07/365)^{3650} = \$4027.24$

d) $B = 2000\,e^{(0.07)(10)} = \$4027.51$

7. a)
$2^{3x} = 2^{x+1}$
$3x = x+1$
$x = \frac{1}{2}$

b)
$e^{3x} = 4$
$3x = \ln(4)$
$x = \ln(4) / 3$

c)
$2\ln(t) = e^0$
$2\ln(t) = 1$
$\ln(t) = \frac{1}{2}$
$t = e^{1/2}$

d)
$x^3 = 2x\ (2)$
$x^2 = 4$
$x = 2$ (Note: can't use negative (-2), as $\ln(x^3)$ is not defined)

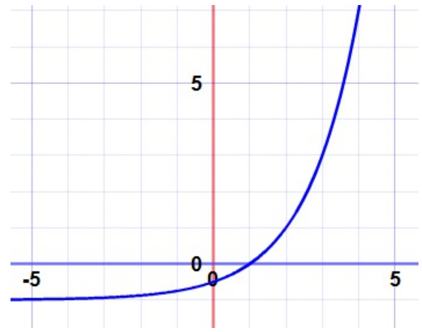
e)
$\log(2) = 4x-2$
$x = 1/4\,(\log(2) + 2)$
$\approx .575$

f)
$7/3 = e^{0.4t}$
$\ln(7/3) = 0.4\,t$
$t = \ln(7/3) / 0.4$
≈ 2.118

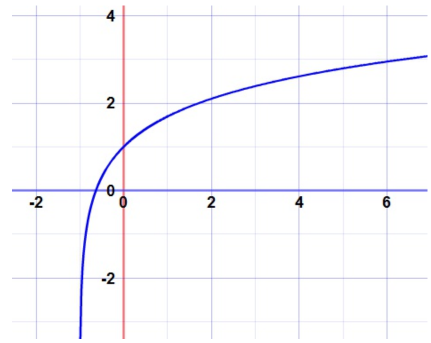
g)
$\ln(x) = e^1$
$\ln(x) = e$
$x = e^e$
≈ 15.154

h)
$\ln(2x+1) + \ln(x) = 2$
$\ln(2x^2 + x) = 2$
$2x^2 + x = e^2$
$2x^2 + x - e^2 = 0$
Use quadratic formula with $a=2$, $b=1$, $c=-e^2$ to find solutions
$x \approx -2.188...$ (discard for ln(x))
and $x \approx 1.688...$ (valid solution)

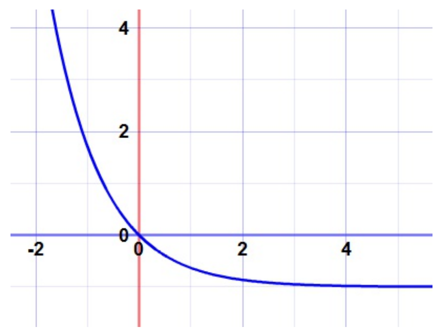
8. a) $y=2^x$ shifted one unit right, one unit down



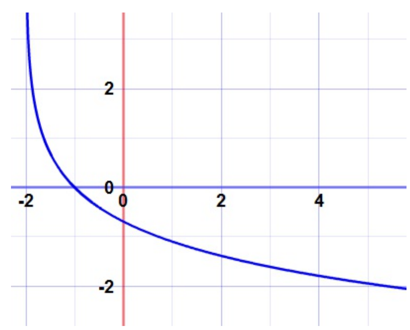
b) $y=\ln(x)$ shifted one left and one unit up



c) $y=e^x$ flipped horizontally and moved down one unit



d) $y=\ln(x)$ shifted two units left, then flipped vertically



Unit 15. Exponential Models

1. Assume the world's population is growing continuously and doubles every 53 years.
   a) Find its annual growth rate.
   b) The population was 6 billion in 1998. Find the population in 2015.
   c) When will the population reach 10 billion?

2. After a chemical spill, the concentration of pollutant in the local water supply is given by $C(t) = 0.1(1+3e^{-0.03t})$, where C is measured in grams per cubic centimetre and t is measured in days.
   a) What is the concentration at the time of the spill (t=0)?
   b) How long after the spill are concentrations below 0.16 g/cm^3?
   c) If this model is correct for all values of t, will C(t) ever return to zero?

3. Radioactive Carbon-14 has a half-life of 5730 years (i.e. the time in which half of the C-14 content in an object will decay). How long will it take for an object to lose 80% of its original C-14?

4. Assume that crude oil consumption grows exponentially at a constant rate. Assume further that in 1940, 10% of all crude oil reserves were used up, and that this number had grown to 40% in 2000.
   a) If trends continue, when would we expect oil reserves to be depleted?
   b) By a miraculous discovery (or perhaps because we decide to invade Mars), the crude oil reserves available to us double! How many additional years at current trends do we gain before oil reserves are depleted now?

5. Due to a persistent bacterial infection, production at a wheat farm has been decreasing exponentially. The yearly harvest can be modelled using a continuous decay model with decay constant k=-0.15. If the farm's output in 2007 was 24 bushels per acre, estimate the farm's output per acre in 2010, assuming current trends continue.

6. With the loss of many of its natural predators, the hare population is growing exponentially in some areas of the province. If the hare population triples every four years, how long would it take for the current hare population to quadruple?

7. Sales at a retail company have been decreasing exponentially ever since a national advertising campaign was cancelled. Sales in 2006 were $62 million, and sales dropped to $51 million in 2007. Assuming that sales will continue to decline at an exponential rate, estimate the 2010 sales for this company.

8. How long will it take an investment to triple in value if it grows at 3% annually, compounded yearly?

Unit 15 - Solutions

1. Growing exponentially, so population $N(t) = N_0 e^{kt}$, where N_0 is starting population.

a) Doubles every 53 years, so when t=53, $N(t) = 2N_0$

Substitute $2N_0 = N_0 e^{53k}$

Solve for: $k = \ln(2)/53$ (≈ 0.013)

The growth rate is about 1.3% per year.

b) Here $N_0=6$ and t=17:

$N(17) = 6 e^{(\ln 2 / 53)(17)} \approx 7.5$ billion

c) Now solve for t such that N(t)=10:

$10 = 6 e^{(\ln 2 /53) t}$

$\ln (5/3) = (\ln(2)/53) t$

Hence $t = \ln(5/3) / (\ln(2)/53) \approx 39$ years , i.e. in 2037.

2.

a) Substitute t=0 to get a concentration of $C(0) = 0.4$ g/cm^3.

b) Solve $0.16 = 0.1 (1 + 3e^{-0.03t})$

$1.6 = 1 + 3e^{-0.03t}$

$0.6 = 3e^{-0.03t}$

$0.2 = e^{-0.03t}$

$\ln (0.2) = -0.03t$ so $t=\ln(0.2)/-0.03 \approx 54$ days.

c) No, since e-x asymptotically approaches zero, even for very large values of t, the concentration will be above $C = 0.1 (1 + 3 \cdot 0) = 0.1$ g/cm^3. If the model is correct for all values of t, concentrations will never fall below this value.

3. C-14 has a half-life of 5730 years, and the initial content is I, then when t=5730, $I(t) = ½ I$.

i.e. $½ I = I e^{5730 k}$, where k is the decay rate.

Solve for $k = \ln (½) / 5730$ ($\approx$-0.00012)

Now, we need 80% of C-14 to be decayed, i.e. we want time find the time t such that $I(t) = 0.2 I$ (since 20% is left):

i.e. $0.2 I = I e^{kt}$ (k as above)

Solve for $t = \ln(0.2) / k \approx 13304$ years.

4. Let P(t) be the consumed percentage of oil reserves. Let t=0 by 1940.

Hence, in t=60 years, P(t) rose from 10% to 40%,

i.e. $40 = 10 e^{60k}$

Solve for the growth rate $k = \ln(4) / 60 \approx 0.023$ (or 2.3 % annual growth)

a) We look for time t such that P(t) = 100%.

Solve $100 = 10 e^{kt}$ (k as above)

i.e. $t = \ln(10)/ k \approx 100$ years.

We would expect oil reserves to be depleted in 2040.

b) We double oil reserves, i.e. have 200% available.

Solve $200 = 10\, e^{kt}$

for $t = \ln(20) / k \approx 130$ years.

A complete doubling of oil reserves only provides an additional 30 years if growth rates remain constant.

5.

The exponential function for the wheat production is given by $f(t)=24\, e^{-0.15t}$, where t is the number of years since 2007.

To estimate production in 2010, we simply need to calculate

$$f(3) = 24\, e^{-0.45}$$

$$\approx 15.3 \text{ bushels/acre}$$

6.

Exponential growth, hence $N(t)= N_0\, e^{kt}$. We need to find k first.

Hare population triples every four years. If current hare population is given by N, then in four years (t=4) the population will be 3N.

I.e. $3N = N\, e^{k4}$

$3 = e^{4k}$

$\ln(3) = 4k$, so $k= \ln(3)/4$ (approximately .274)

Now we wish to solve for time t such that population has grown to 4N,

I.e. $4N = N\, e^{kt}$

$4 = e^{kt}$

$t = \ln(4)/k = \ln(4)/(\ln(3)/4) \approx 5.05$ years.

The population will quadruple (almost) every five years.

7.

Exponential decay, hence sales $S(t)= 62\, e^{kt}$, where t is years after 2006.

We need to find k first. (Will be negative, given decay).

In one year (t=1), sales dropped to 51 million,

i.e. $51 = 62\, e^{k}$

So $k = \ln(51/62)$ (approximately -.195)

Now estimate 2010 (t=4) sales:

$S(4) = 62\, e^{\ln(51/62)4} \approx 28.4$ million.

Sales will fall to 28.4 million in 2010.

8.

We want our initial investment I to triple (3I) at 3% yearly, i.e.

Solve for t: $3\, I = I\, (1.03)^{t}$

$\ln(3) = \ln(1.03\, t)$

$\ln(3) = t \ln(1.03)$ so $t = \ln(3)/\ln(1.03) \approx 37$ years.

Unit 16. Trigonometry

1. Convert each angle to radian measure:
   a) $\theta=120^{o}$ b) $\theta=270^{o}$ c) $\theta=-420^{o}$

2. Convert each angle to degrees:
   a) $-3\pi/4$ b) $4\pi/3$ c) 3π

3. Use the unit circle to find the exact answer in each case:
   a) $\sin(5\pi/6)$ b) $\cos(7\pi/6)$ c) $\tan(-2\pi/3)$

4. Suppose for an angle "x" (in radian measure) we know that $\sin(x)=A$ and $\cos(x)=B$.
   Evaluate (in terms of A and B):
   a) $\sin(\pi-x)$ b) $\sin(x+\pi/2)$ c) $\tan(\pi+x)$

5. A surveyor is stationed 130m away from the base of a tall building. She measures the angle of elevation to the top of the building, and finds it to be 51.2^{0}. Find the building's height.

6. A ladder is leaning against the side of a building. If the ladder forms an angle of 60^{o} with the ground, and the foot of the ladder is 10 metres away from the base of the wall, how long is the ladder?

7. Solve for x (i.e. find all angles x that satisfy the given equations):
   a) $\cos(x)=\sqrt{3}/2$ b) $\sin(x)=-1$ c) $\sin(x)=-½$ d) $\tan(x)=-1$

8. Find the domain of the function

   a) $f(x)=\dfrac{x}{1-2\sin(x)}$ b) $f(x)=\dfrac{1}{2\sin^2(x)+\cos(x)-1}$

9. Find the x-intercepts of the function $f(x)=1+\cos(x-\pi)$ in two ways:
   a) Use the trigonometric identities to solve $f(x)=0$ algebraically.
   b) Sketch a graph of $f(x)$, by shifting/scaling the basic graph of $y=\cos(x)$.

10. Solve the inequality $2\sin(x)+4<3$

Unit 16 - Solutions

1. Recall that $\pi = 180^o$

   a) $2\pi/3$ b) $3\pi/2$ c) $-7\pi/3$

2. a) -135^o b) 240^o c) 540^o

3. a) $\sin(\pi/6) = ½$, so $\sin(5\pi/6) = ½$ as well.

   b) $\cos(\pi/6) = \sqrt{3}/2$ so $\cos(7\pi/6) = -\sqrt{3}/2$

   c) $\tan(x) = \sin(x)/\cos(x)$.
   $\sin(\pi/3) = \sqrt{3}/2$ so $\sin(-2\pi/3) = -\sqrt{3}/2$
   $\cos(\pi/3) = ½$ so $\cos(-2\pi/3) = -½$
   combine to find $\tan(-2\pi/3) = (-\sqrt{3}/2) / (-1/2) = \sqrt{3}$

4. (To solve these, draw a unit circle with any angle 'x',
   and label vertical (=sin(x)) as A and horizontal (=cos(x)) as B.)

   a) $(\pi-x)$ is the angle "x" away from π,
   so $\sin(\pi-x) = A$ as well.
   b) $(x+\pi/2)$ is 90^o further than "x".
   so $\sin(x+\pi/2) = \cos(x) = B$
   c) $(-x)$ is the angle opposite "x"
   so $\tan(-x) = \sin(-x)/\cos(-x) = (-A)/(-B) = A/B$

5. Since $\tan(\theta)$ = opposite/adjacent,
   we have $\tan(51.2^o)$ = height / 130m
   i.e. height = $130 \tan(51.2^o) \approx 161.7$ metres

6. The ladder's length is the hypotenuse, the bottom side is adjacent to the angle of 60^o.
   We can use the cosine, since cos(x) = adjacent / hypotenuse.
   So hypotenuse = adjacent / cos(x)
   = 10 metres / $\cos(\pi/3)$
   = 10 / ½
   = 20 m. The ladder is 20m long.

7. a) $x = \pi/6 + 2k\pi$ and $x = 11\pi/6 + 2k\pi$, $k \in I$
   b) $x = 3\pi/2 + 2k\pi$, $k \in I$
   c) $x = 7\pi/6 + 2k\pi$ and $x = 11\pi/6 + 2k\pi$, $k \in I$
   d) $x = 3\pi/4 + 2k\pi$ and $x = 7\pi/4 + 2k\pi$, $k \in I$
   (or combine as just $x = 3\pi/4 + k\pi$, $k \in I$)

8. a) We cannot divide by zero, i.e. $1 - 2\sin(x) = 0$ is not allowed.

Solve $1 = 2\sin(x)$, i.e. $\sin(x)=½$,

for $x=\pi/6 + 2k\pi$ and $x=5\pi/6 + 2k\pi$, $k \in I$

b) Again, we need to solve $2\sin^2(x) + \cos(x) - 1 = 0$

Use trig identities: $2(1-\cos^2(x)) + \cos(x) - 1 = 0$

$2\cos^2(x) - \cos(x) - 1 = 0$

Factor: $(2\cos(x) + 1)(\cos(x) - 1) = 0$

Now solve both $\cos(x) = -1/2$ and $\cos(x) = 1$

for $x=2\pi/3 + 2k\pi$ and $x = 0 + 2k\pi$

$x=4\pi/3 + 2k\pi$

Combine: This function is not defined at $x=0$, $x=2\pi/3$, $x=4\pi/3$ (all $+2k\pi$, $k\in I$)

9. a) We want to solve $0 = 1 + \cos(x-\pi)$

Use addition forumla: $0 = 1 + \cos(x)\cos(-\pi) - \sin(x)\sin(-\pi)$

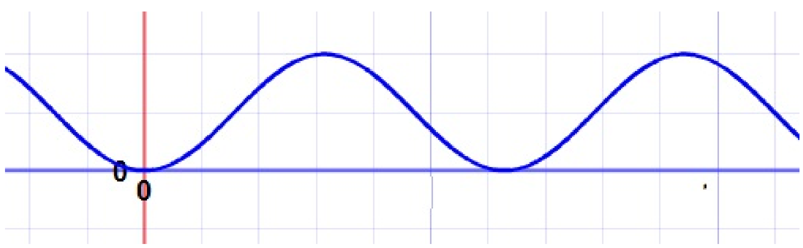
$0 = 1 + \cos(x)(-1) - \sin(x)(0)$

$0 = 1 - \cos(x)$

$\cos(x) = 1$

Hence intercepts are $x=0+2k\pi$, $k\in I$

b) Graph (shift $\cos(x)$ to the right by π units and up by 1 unit):



We see that x-intercepts occur at $x=0$ and every 2π before/after.

10. Solve $\sin(x)<-½$

for $7\pi/6 < x < 11\pi/6$ $(+ 2k\pi,\ k \in I)$

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Version 1.3, 3 November 2008

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