Mathematics 110 (Calculus I) Laboratory Manual

Department of Mathematics & Statistics
University of Regina

5^{th} edition

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History

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Contents

Introduction

"*One does not learn how to swim by reading a book about swimming,*" as surely everyone agrees. The same is true of mathematics. One does not learn mathematics by only reading a textbook and listening to lectures. Rather, one learns mathematics by doing mathematics.

This Laboratory Manual is a set of problems that are representative of the types of problems that students of Mathematics 110 (Calculus I) at the University of Regina are expected to be able to solve on quizzes, midterm exams, and final exams. In the weekly lab of your section of Math 110 you will work on selected problems from this manual under the guidance of the laboratory instructor, thereby giving you the opportunity to do mathematics with a coach close at hand. These problems are not homework and your work on these problems will not be graded. However, by working on these problems during the lab periods, and outside the lab periods if you wish, you will gain useful experience in working with the central ideas of elementary calculus.

The material in the Lab Manual does not replace the textbook. There are no explanations or short reviews of the topics under study. Thus, you should refer to the relevant sections of your textbook and your class notes when using the Lab Manual. These problems are not sufficient practice to master calculus, and so you should solidify your understanding of the material by working through problems given to you by your professor or that you yourself find in the textbook.

To succeed in calculus it is imperative that you attend the lectures and labs, read the relevant sections of the textbook carefully, and work on the problems in the textbook and laboratory manual. Through practice you will learn, and by learning you will succeed in achieving your academic goals. We wish you good luck in your studies of calculus.

Module 1

Equations and Functions

1.1 Solving Equations

1-10: Solve the given equations.

Answers: Page 40

1. $x^2 - 6x + 9 = 0$
2. $2x^2 - 5x - 3 = 0$
3. $4x^2 + 3x + 1 = 0$
4. $x^3 - 2x - 4 = 0$
5. $x^3 - 4x^2 - 4x + 16 = 0$
6. $x^3 + 4x - 5 = 0$
7. $2x^4 - 3x^2 = 0$
8. $x^4 - 3x^2 + 2 = 0$
9. $3x^5 - 2x^3 + 3x^2 - 2 = 0$
10. $2x^5 + 5x^4 - 3x^2 = 0$

1.2 Properties of Functions: Domains, Intercepts, Symmetry

1-16: Find the domain and the x- and y-intercepts (if there are any) of the given functions.

Answers: Page 40

1. $f(x) = x^3$
2. $h(x) = \sqrt{x-6}$
3. $f(x) = x^2 + 4x + 4$
4. $g(x) = x^4 + 4x^3 - 5x^2$
5. $f(x) = \dfrac{1}{x-1}$
6. $h(x) = \dfrac{1}{x^2 + 4x + 4}$
7. $p(x) = \sqrt{4 - x^2}$
8. $f(x) = 3x^2 + 5x + 2$
9. $g(x) = x^3 + 3x - 4$
10. $h(x) = \dfrac{x+5}{x+7}$
11. $f(x) = \dfrac{10}{2x^2 - 5x - 3}$
12. $g(x) = \sqrt{4 - x}$
13. $p(x) = \sqrt{x^2 - 10}$
14. $h(x) = \sqrt{\dfrac{x+6}{2x-3}}$
15. $f(x) = \dfrac{\sqrt{x^2 - 10}}{x^2 + 10}$
16. $f(x) = \dfrac{1}{\sqrt{x+2}} - \dfrac{1}{x}$

17-32: Determine whether each of the given functions is even, odd or neither.

17. $f(x) = 3x^2$
18. $g(x) = 2x^3$
19. $h(x) = 3x^2 + 2x^3$
20. $f(t) = -6t^3$
21. $g(u) = u^3 - u^2$
22. $f(x) = x^3 + x^5$
23. $f(x) = \sqrt{4 - x^2}$
24. $h(x) = \dfrac{x^2 + 1}{\sqrt{4 - x^2}}$
25. $f(x) = 5x^2 + 3$
26. $g(t) = 4t^3 + t$
27. $h(z) = z^5 + 1$
28. $f(t) = \sqrt{t^2 + 5}$
29. $g(x) = \dfrac{x}{x^4 + 3}$
30. $h(u) = \dfrac{3 + u^2}{u + 1}$
31. $p(x) = \dfrac{x^5}{x^3 + x}$
32. $u(s) = \sqrt{\dfrac{s^6 + 4}{s^{10} + 7}}$

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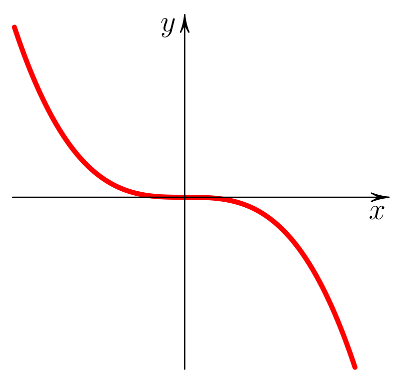
33. Here is a graph of a function f:



(a) Sketch the graph of $-f(x)$.
(b) Sketch the graph of $f(-x)$.
(c) Sketch the graph of $f(x + 1)$.
(d) Sketch the graph of $f(x) + 1$.
(e) Is f even, odd, or neither?

1.3 Function Algebra

1. Suppose that $f(x) = \dfrac{1}{x+2}$. Determine:

   (a) $f(x+2)$

   (b) $f(f(x))$

Answers: Page 42

2-5: Find the composite functions $f \circ g$ and $g \circ f$ and their domains for the given functions.

2. $f(x) = 2x^3$, $g(x) = \sqrt{x^2+3}$.

3. $f(x) = 3x^2 + 6x + 4$, $g(x) = 3x - 2$

4. $f(z) = \sqrt{z^2+5}$, $g(z) = \dfrac{z}{z+1}$

5. $f(x) = \dfrac{2x+5}{x-4}$, $g(x) = x^2 + 3$

---

6. Find two functions $f(x)$ and $g(x)$ such that $f(g(x)) = \sqrt{x^2+1} - 3$.

7. Use function composition to solve the equation $\left(x^2-5\right)^2 + 7x^2 - 23 = 0$. (Hint: Note the equation can be rewritten as $\left(x^2-5\right)^2 + 7\left(x^2-5\right) + 12 = 0$.)

Module 1 Review Exercises

1-5: Solve the given equations.

Answers: Page 42

1. $2x^2 + 3x - 2 = 0$
2. $x^2 + x - 20 = 0$
3. $x^3 + 2x^2 - x - 2 = 0$
4. $x^4 - 5x^2 + 4 = 0$
5. $x^5 - 4x^3 - x^2 + 4 = 0$

6-9: Find the domain and the x- and y-intercepts of the given functions.

6. $h(x) = \dfrac{2x-3}{5x+4}$
7. $f(x) = \sqrt{2x^2 - 8}$
8. $g(x) = \sqrt{\dfrac{2x+1}{x+5}}$
9. $p(x) = \dfrac{\sqrt{x+8}}{x-3}$

10-13: Determine whether each of the given functions is even, odd or neither.

10. $f(x) = \dfrac{x^4+3}{x^2+1}$
11. $g(t) = t^{2/3}$
12. $h(z) = z\sqrt{z^2+1}$
13. $g(x) = \dfrac{x^3}{x^6+5} - x$

14-16: Find the composite functions $f \circ g$ and $g \circ f$ and their domains.

14. $f(x) = x^3 + 6$, $g(x) = x^{2/3}$
15. $f(t) = \dfrac{2t+5}{t-4}$, $g(t) = t^2 + 3$
16. $f(x) = \sqrt{x-1}$, $g(x) = \dfrac{x+5}{x+3}$

Module 2

Limits

2.1 Secant and Tangent Lines

1. Consider the curve described by the function $y = x^3 + x^2 - 2x + 3$.

   (a) Show that the points $(-1, 5)$ and $(0, 3)$ lie on the curve.

   (b) Determine the slope of the secant line passing through the points $(0, 3)$ and $(-1, 5)$.

   (c) Let Q be the arbitrary point $(x, x^3 + x^2 - 2x + 3)$ on the curve. Find the slope of the secant line passing through Q and $(-1, 5)$.

   (d) Use your answer in (c) to determine the slope of the tangent line to the curve at the point $(-1, 5)$.

Answers: Page 43

2.2 Calculating Limits Using Limit Laws

1-15: Evaluate the following limits.

Answers: Page 43

1. $\displaystyle\lim_{x\to 3} \frac{x^2+4}{x+2}$

2. $\displaystyle\lim_{x\to -2} \frac{x^2+4x+4}{x+2}$

3. $\displaystyle\lim_{x\to 0} \frac{2-\sqrt{4-x}}{x}$

4. $\displaystyle\lim_{x\to 3} |x-3|$

5. $\displaystyle\lim_{x\to -2} \frac{x^2-3x-10}{x^2-4}$

6. $\displaystyle\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$

7. $\displaystyle\lim_{t\to 1} \frac{t-\sqrt{t}}{\sqrt{t}-1}$

8. $\displaystyle\lim_{x\to 1} \frac{x^3-2x+1}{x-1}$

9. $\displaystyle\lim_{t\to 4} \frac{\sqrt{t}-2}{t-4}$

10. $\displaystyle\lim_{y\to 3} \frac{2-\sqrt{y^2-5}}{y^2-y-6}$

11. $\displaystyle\lim_{x\to -4} \frac{(x+1)^2-9}{x+4}$

12. $\displaystyle\lim_{u\to -5} \frac{\frac{1}{u}+\frac{1}{5}}{u+5}$

13. $\displaystyle\lim_{t\to 2} \frac{t^2+t-6}{\frac{2}{t}-1}$

14. $\displaystyle\lim_{x\to 4} \frac{x^3-4x^2-4x+16}{x-4}$

15. $\displaystyle\lim_{t\to 1} \frac{t^3+4t-5}{t^3-1}$

2.3 Trigonometric Limits

1-11: Evaluate the trigonometric limits.

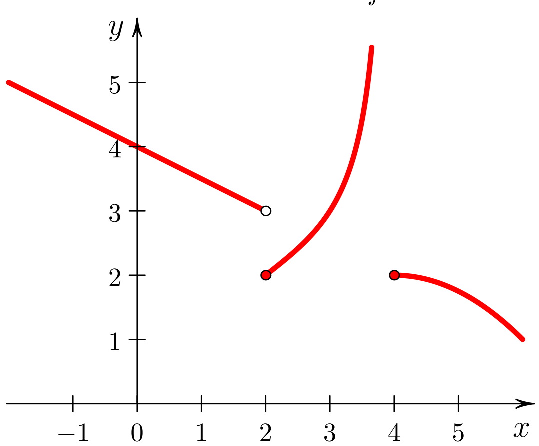
Answers: Page 43

1. $\lim\limits_{\theta\to 0} \dfrac{\theta}{\sin\theta}$
2. $\lim\limits_{\theta\to 0} \theta\cot\theta$
3. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x}$
4. $\lim\limits_{\theta\to 0} \dfrac{7\theta}{\sin 5\theta}$
5. $\lim\limits_{\theta\to 0} \dfrac{7\theta}{\cos 5\theta}$
6. $\lim\limits_{x\to 0} \dfrac{\sin x}{x\cos x}$
7. $\lim\limits_{x\to \pi} \dfrac{\cos x}{x}$
8. $\lim\limits_{\theta\to \frac{\pi}{4}} \dfrac{\tan\theta}{\theta}$
9. $\lim\limits_{t\to \frac{\pi}{2}} \dfrac{\sin t - 1}{\cos t}$
10. $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$
11. $\lim\limits_{\theta\to \pi} \dfrac{\cos\theta + 1}{\sin^2\theta}$

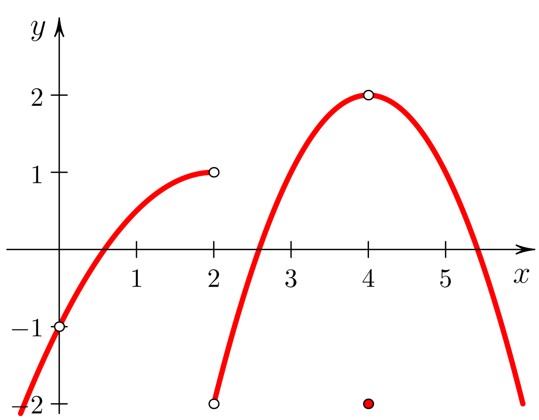
2.4 One-sided Limits

1. Find the one-sided limits of f at the values $x = 0$, $x = 2$ and $x = 4$.

Answers: Page 43



2. Find the one-sided and two-sided limits of f at the values $x = 0$, $x = 2$ and $x = 4$.



3. Suppose $f(x) = \begin{cases} \frac{4}{x+4} & \text{if } x < 2 \\ x^2+1 & \text{if } x \geq 2 \end{cases}$.

   Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to 2} f(x)$, if they exist.

4. Suppose $f(x) = \begin{cases} x^2+2cx & \text{if } x \leq -1 \\ x+5c & \text{if } x > -1 \end{cases}$.

   Find all values for the constant c that make the two-sided limit exist at $x = -1$.

2.5 Infinite Limits and Vertical Asymptotes

1-4: Determine the following limits. For any limit that does not exist, identify if it has an infinite trend (∞ or $-\infty$).

Answers: Page 44

1. $\lim_{x\to 2^+} \dfrac{5x+4}{2x-4}$
2. $\lim_{x\to 3^-} \dfrac{x^2+2x}{x^2-5x+6}$
3. $\lim_{x\to 5} \dfrac{x^2-4x-5}{x^2-3x-10}$
4. $\lim_{x\to 0} \dfrac{\sec x}{x^2}$

5-12: Find the vertical asymptotes of the following functions.

5. $f(x) = \dfrac{3x+3}{2x-4}$
6. $f(x) = x^3+5x+2$
7. $g(t) = \dfrac{\sqrt{t^2+3}}{t-2}$
8. $f(x) = \dfrac{x^2-2x+1}{2x^2-2x-12}$
9. $f(x) = \dfrac{\cos x}{x}$
10. $y = \dfrac{5x^2-3x+1}{x^2-16}$
11. $f(x) = \dfrac{x^3+1}{x^3+x^2}$
12. $F(x) = \dfrac{x}{\sqrt{4x^2+1}}$

2.6 Continuity

1. Define precisely what is meant by the statement "f is continuous at $x = a$".

2-7: Use the continuity definition to determine if the function is continuous at the given value. If the function is discontinuous there, decide whether it is a removable, jump, or infinite discontinuity.

Answers: Page 44

2. $f(x) = x^3+5x+1$, at $x = 2$
3. $g(t) = \dfrac{t+1}{t^2+4}$, at $t = -1$
4. $h(y) = \dfrac{y^2+4y+4}{y+2}$, at $y = -2$

5. $p(s) = \sqrt{s} - 4$, at $s = 2$

6. $f(x) = \begin{cases} x^2+1, & \text{if } x \leq 1 \\ \frac{x+1}{x-1}, & \text{if } x > 1 \end{cases}$, at $x = 1$

7. $g(t) = \begin{cases} 2t+3, & \text{if } t \leq 2 \\ \frac{t^2-5t+6}{t-2}, & \text{if } t > 2 \end{cases}$, at $t = 2$

---

8. Let c be a constant real number and f be the function

$$f(x) = \begin{cases} \sqrt{-x}+1 & \text{if } x < 0 \\ x^2 + c^2 & \text{if } x \geq 0 \end{cases}$$

   (a) Explain why, for $c = -2$, the function f is discontinuous at $x = 0$.

   (b) Determine all real numbers c for which f is continuous at $x = 0$.

9. Where is the function $f(x) = \dfrac{x^2+3x+2}{x^2-1}$ continuous?

10. Using the Intermediate Value Theorem, show there is a real number c strictly between 1 and 3 such that $c^3 + 2c^2 = 10$.

11. Show that the equation $x^2 + \cos x - 2 = 0$ has a solution in the interval $(0, 2)$.

Module 2 Review Exercises

Answers: Page 45

1-5: Evaluate the limits.

1. $\lim\limits_{x\to 3} \dfrac{2x^2-5x-3}{x^2+x-12}$

2. $\lim\limits_{t\to 4} \dfrac{\sqrt{t}-2}{t^2-2t-8}$

3. $\lim\limits_{x\to -1} \dfrac{x^3+x^2+2x+2}{x^2-2x-3}$

4. $\lim\limits_{t\to -2} \dfrac{\sqrt{10+3t}-2}{3t^2+4t-4}$

5. $\lim\limits_{t\to 2} \dfrac{\frac{8}{t}-4}{3t^2-4t-4}$

6-9: Evaluate the trigonometric limits.

6. $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{\sin(5x)}$

7. $\lim\limits_{\theta\to 0} \dfrac{\sin(3\theta)}{\tan(4\theta)}$

8. $\lim\limits_{t\to \pi} \dfrac{2\sin^2 t}{1+\cos t}$

9. $\lim\limits_{x\to 0} \dfrac{\cos(3x)+\cos(4x)-2}{x}$

10-12: Determine whether the functions are continuous at the given value.

10. $f(x) = \dfrac{x+3}{\sqrt{x^2+5}}$ at $x=-1$

11. $h(t) = \dfrac{t^2+2t-1}{t-3}$ at $t=3$

12. $g(x) = \begin{cases} 3x^2-1, & \text{if } x \le 2 \\ \frac{x^2+x-6}{x-2}, & \text{if } x > 2 \end{cases}$ at $x=2$

Module 3

Differentiation

3.1 Average and Instantaneous Rate of Change

1. Consider the function $f(x) = x^3 + x + 2$.

Answers: Page 45

   (a) Find the slope of the secant line through points $(2, 12)$ and $(1, 4)$ on the graph of f.

   (b) From your graph, estimate the slope of the tangent line to the curve at the point $(2, 12)$. Next, numerically estimate the tangent slope by calculating the secant slope between the point $(2, 12)$ and a point with x value near 2.

   (c) What is the equation of the tangent line at $(2, 12)$? (Use your estimate from (b) for the slope.)

2. After t seconds, a toy car moving along a straight track has position $s(t)$ measured from a fixed point of reference given by $s(t) = t^3 + 2t^2 + 1$ cm.

   (a) How far is the car initially from the reference point?

   (b) How far from the reference point is the car after 2 seconds?

   (c) What is the average velocity of the car during its first 2 seconds of motion?

   (d) By calculator, estimate the instantaneous velocity of the car at time $t = 2$ by computing the average velocities over small time intervals near $t = 2$.

3. One mole of an ideal gas at a fixed temperature of 273 K has a volume V that is inversely proportional to the pressure P (Boyle's Law) given by

$$V = \frac{22.4}{P} \ ,$$

   where V is in litres (L) and P is in atmospheres (atm).

   (a) What is the average rate of change of V with respect to P as pressure varies from 1 atm to 3 atm?

   (b) Use a calculator to estimate the instantaneous rate of change in the volume when the pressure is 3 atm by computing the average rates of change over small intervals lying to the left and right of $P = 3$ atm.

3.2 Definition of Derivative

1-2: Use the definition of the derivative at a value a to find the following.

Answers: Page 45

1. $f'(2)$ if $f(x) = \dfrac{1}{x+1}$

2. $g'(4)$ if $g(x) = \sqrt{2x}$

3-8: Use the definition of the derivative to calculate $f'(x)$ for each of the following functions.

3. $f(x) = x^2 + 3$

4. $f(x) = \dfrac{1}{3x}$

5. $f(x) = (x+2)^2$

6. $f(x) = \sqrt{x+2}$

7. $f(x) = \dfrac{3x+2}{x+1}$

8. $f(x) = \dfrac{1}{\sqrt{x}}$

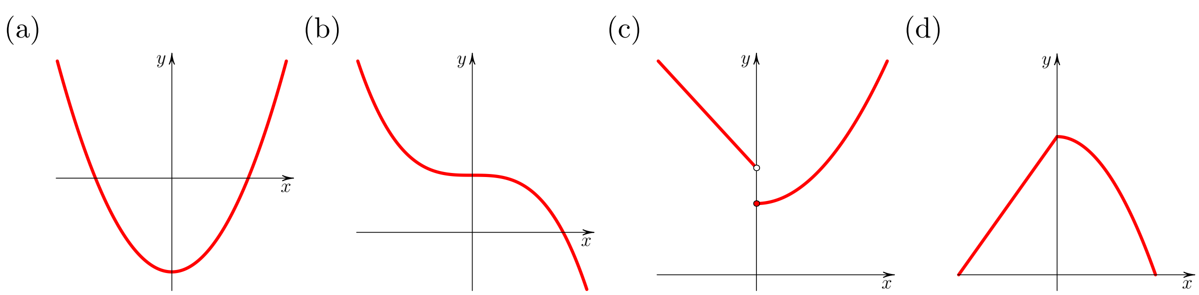
---

9. Prove that $f(x) = \sqrt{(x-2)^2}$ is not differentiable at $x = 2$ by showing that the following left and right hand limits differ:
$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} \neq \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h}.$$

3.3 Derivative Function

1. Which of the following graphs represent functions that are differentiable at $x = 0$? (Explain why or why not).

Answers: Page 46



3.4 Differentiation Rules: Power, Sums, Constant Multiple

1-8: Differentiate the following functions involving powers, sums and constant multiplication. (Any value that is not the function variable should be considered a constant.)

Answers: Page 46

1. $f(x) = \sqrt{x} - x^{12}$
2. $g(x) = \dfrac{1}{\sqrt{x^5}}$
3. $y = \dfrac{4x^3}{5}$
4. $f(u) = u^{-4} + u^4$; Also find $f'(1)$.
5. $f(x) = (x^3 + x)^2$ (Hint: Expand first.)
6. $f(x) = \sqrt{3x} + \sqrt[5]{\dfrac{x}{3}}$

   (Hint: $\sqrt[n]{xy} = (\sqrt[n]{x})(\sqrt[n]{y})$ and $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$.)
7. $f(x) = \sin(\pi/15)x^{2a}$
8. $s(t) = -\dfrac{g}{2}t^2 + v_0 t + s_0$

---

9. Calculate the instantaneous rates of change given in problems 1(b), 2(d), and 3(b) of Section 3.1 directly using the derivative.
10. Find the equation of the tangent line to the curve $y = x + \sqrt{x}$ at the point $P(1, 2)$.
11. Find the value(s) of x for which the curve $y = 2x^3 - 4x^2 + 5$ has a horizontal tangent line.

3.5 Rates of Change

1-2: The following problems consider the meaning of the derivative as a rate of change.

Answers: Page 46

1. The concentration of carbon dioxide in the Earth's atmosphere has been observed at Mauna Loa Observatory in Hawaii to be steadily increasing (neglecting seasonal oscillations) since 1958. A best fit curve to the data measurements taken from 1982 to 2009 yields the following function for the concentration in parts per million (ppm) as a function of the year t:

$$C(t) = 0.0143(t - 1982)^2 + 1.28(t - 1982) + 341$$

   (a) What was the level of CO_2 in the air in the year 2000?
   (b) At what rate was the CO_2 level changing with respect to time in the year 2000?
   (c) By what percentage did the CO_2 level change between 2000 and 2005?

2. A conical tank has a height of 5 metres and radius at the top of 2 metres.

   (a) Show that the volume of liquid in the tank when it is filled to a depth y is given by

$$V = \frac{4\pi}{75} y^3$$

   (b) What is the rate of change of volume with respect to depth when the tank is filled to 4 metres?

3.6 Differentiation Rules: Products and Quotients

Answers: Page 47

1-8: Differentiate the following functions involving products and quotients. (Any value that is not the function variable should be considered a constant.)

1. $f(x) = \left(x^4 - 3x^2 + 2\right)\left(x^{\frac{1}{3}} - x\right)$
2. $h(x) = \left(x^3 + \pi x + 2\right)\left(2 + \dfrac{1}{x^3}\right)$
3. $y = \left(x^2 - 1\right)\left(x^3 + 2\right)\left(2x^2 + \sqrt{x}\right)$
4. $f(x) = \dfrac{x-4}{x-6}$
5. $f(\theta) = \dfrac{\theta^2 + 3\theta - 4}{\theta^2 - 7}$
6. $g(x) = \dfrac{1+x^2}{\sqrt{x}}$; Also find $g'(4)$.
7. $f(v) = \dfrac{(2v+3)(v+4/v)}{v^2+v}$
8. $h(x) = cx^2 + \left(3\sqrt{x} + 2\right)\left(2x^2 + x\right)$

3.7 Parallel, Perpendicular, and Normal Lines

Answers: Page 47

1. Find the line through the point $P(2,1)$ that is parallel to the tangent to the curve $y = 3x^2 + 2x + 1$ at the point $Q(1,6)$.
2. Find the normal line to the curve $y = \sqrt{x} + x^2$ at the point $P(1,2)$.
3. Find any points on the curve $y = x^3 - 4$ with normal line having slope $-\dfrac{1}{12}$.

3.8 Differentiation Rules: General Power Rule

Answers: Page 47

1-6: Differentiate the following functions requiring use of the General Power Rule. (Any value that is not the function variable should be considered a constant.)

1. $f(x) = \left(x^2 + 3\right)^9$
2. $g(x) = \dfrac{1}{x + \sqrt{x}}$
3. $f(t) = \dfrac{7}{\sqrt{2t^2 + 3t + 4}}$
4. $y = \left(\dfrac{4x+3}{x^2+x}\right)^{-\frac{1}{7}}$
5. $h(x) = \sqrt[3]{5x^n + 4c}$
6. $f(x) = \left[\left(2x + \sqrt{x}\right)^4 + 3x\right]^5$

3.9 Differentiation Rules: Trigonometric Functions

Answers: Page 48

1. Find the derivative $f'(x)$ of $f(x) = \sin 4x$ using the definition of the derivative.

2-5: Differentiate the following functions involving trigonmetric functions.

2. $f(x) = x^2 \cos x$
3. $f(t) = \dfrac{t^3}{\sin t + \tan t}$
4. $H(\theta) = \csc\theta \cot\theta$; Also find $\left.\dfrac{dH}{d\theta}\right|_{\theta=\pi/3}$.
5. $f(x) = (\sin x + \cos x)(\sec x - \cot x)$

---

6. Calculate $\dfrac{df}{d\theta}$ for the function $f(\theta) = \sin^2\theta + \cos^2\theta$
   (a) Directly by using the rules of differentiation.
   (b) By first simplifying f with a trigonometric identity and then differentiating.

7. For the curve $y = x\tan x$ and point $P\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$,
   (a) Confirm the point P lies on the curve. (b) Find the equation of the tangent line at P.

3.10 Differentiation Rules: Chain Rule

1-13: Differentiate the following functions requiring use of the Chain Rule. (Any value that is not the function variable should be considered a constant.)

Answers: Page 48

1. $f(x) = \left(x^8 - 3x^4 + 2\right)^{12}$
2. $g(x) = \sqrt{3x^2 + 2}$; Also find $g'(2)$.
3. $f(\theta) = \sin(\theta^2)$
4. $h(\theta) = \cot^2\theta$
5. $f(x) = \sec\left[\left(x^3 + 3\right)\left(\sqrt{x} + x\right)\right]$
6. $y = 4\cos\sqrt[3]{x}$
7. $f(x) = \dfrac{1}{3 + \sin^2 x}$
8. $y = (\csc x + 2)^5 + x^2 + x$
9. $y = \pi\tan\theta + \tan(\pi\theta)$
10. $g(x) = \left(\sqrt{x + \sqrt{x}}\right)\left(x^4 - 1\right)^7$
11. $f(x) = \left(\dfrac{x-3}{x+1}\right)^3$
12. $A(t) = \cos(\omega t + \phi)$; Also find $\left.\dfrac{dA}{dt}\right|_{t=0}$.
13. $f(x) = \sin\left[\cos\left(x^2 + x\right)\right]$

---

14. For the curve $y = 5x + 3\sin(2x) - 2\cos(3x)$ and point $P(0, -2)$,
    (a) Confirm the point P lies on the curve. (b) Find the equation of the tangent line at P.
15. Find the value(s) of θ for which the curve $f(\theta) = \cos^2\theta - \sin\theta$ has a horizontal tangent line.

3.11 Differentiation Rules: Implicit Differentiation

1-6: Calculate y' for functions $y = y(x)$ defined implicitly by the following equations. (Any value that is not x or y should be considered a constant.)

Answers: Page 49

1. $x^2 + y^2 = 3x$
2. $xy^2 - 2x^3y + x^3 = 1$; Also find $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)}$.
3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
4. $\sin(xy) = y$

5. $\cos(x+y)+x^2=\sin y$

6. $\dfrac{\sec y}{x}=\sin x$

---

7. Consider the curve generated by the relation $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4$.

   (a) Confirm that the point $P(-1,3\sqrt{3})$ lies on the curve.

   (b) Find the equation of the tangent line to the curve at the point P .

3.12 Differentiation Using All Methods

1-12: Differentiate the following functions using any appropriate rules.

Answers: Page 49

1. $f(x)=4x^5+3x^2+x+4$

2. $y=x^8+\dfrac{2}{x^3}-\sqrt{x}+\dfrac{4}{\sqrt{x}}+10$

3. $g(t)=\sqrt[3]{t^2}+t^4+\dfrac{6}{t^2}$

4. $f(x)=\dfrac{x+5}{\sqrt{x}}$

5. $g(x)=\left(\sqrt{x}+3x+1\right)(x+\pi)$

6. $h(y)=\dfrac{(y+4)^3}{y+5}$

7. $y=\sqrt[4]{x^3+2x+5}$

8. $f(\theta)=\cos 3\theta+\sin^2\theta$

9. $g(x)=\tan\left(x^2+1\right)\cos(x)$

10. $y=\sqrt[4]{(\sin t+5)^3}$

11. $f(x)=\sqrt[3]{\dfrac{x^4+5x-1}{x^2-3}}$

12. $x^3y^4+y^2=xy+6$

3.13 Higher Derivatives

1-4: Calculate the second derivative for each of the following functions.

Answers: Page 50

1. $f(x)=\cot x$

2. $f(x)=(x-2)^{10}$; Also find $f''(3)$.

3. $y=x^3\sec x$

4. $x^2-y^2=16$ (Use implicit differentiation.)

5-6: When one uses derivatives, their simplification becomes important.

5. Show that $f''(x)=\dfrac{32\left(3x^2+16\right)}{\left(x^2-16\right)^3}$ for $f(x)=\dfrac{x^2}{x^2-16}$.

6. Show that $f''(x)=\dfrac{8x+8}{(x-2)^4}$ for $f(x)=\dfrac{x^2}{x^2-4x+4}$.

3.14 Related Rates

1-7: Solve the following problems involving related rates.

Answers: Page 50

1. An oil spill spreads in a circle whose area is increasing at a constant rate of 10 square kilometres per hour. How fast is the radius of the spill increasing when the area is 18 square kilometres?
2. A spherical balloon is being filled with water at a constant rate of 3 cm^3/s. How fast is the diameter of the balloon changing when it is 5 cm in diameter?
3. An observer who is 3 km from a launchpad watches a rocket that is rising vertically. At a certain point in time the observer measures the angle between the ground and her line of sight of the rocket to be $\pi/3$ radians. If at that moment the angle is increasing at a rate of 1/8 radians per second, how fast is the rocket rising when she made the measurement?
4. A water reservoir in the shape of a cone has height 20 metres and radius 6 metres at the top. Water flows into the tank at a rate of 15 $\mathrm{m}^3/\min$, how fast is the level of the water increasing when the water is 10 m deep? Hint: Use similar triangles.
5. At 8 a.m., a car is 50 km west of a truck. The car is traveling south at 50 km/h and the truck is traveling east at 40 km/h. How fast is the distance between the car and the truck changing at noon?
6. A boy is walking away from a 15-metres high building at a rate of 1 m/sec . When the boy is 20 metres from the building, what is the rate of change of his distance from the top of the building?
7. The hypotenuse of a right angle triangle has a constant length of 13 cm. The vertical leg of the triangle increases at the rate of 3 cm/sec. What is the rate of change of the horizontal leg, when the vertical leg is 5 cm long?

3.15 Differentials

1-2: Find the volume V and the absolute error ΔV of the following objects.

Answers: Page 51

1. A cubical cardboard box with side length measurement of $l = 5.0 \pm 0.2$ cm.
2. A spherical cannonball with measured radius of $r = 6.0 \pm 0.5$ cm.

3-4: Find the linear approximation (linearization) $L(x)$ of the function at the given value of x.

3. $f(x) = \sqrt{2x^3 - 7}$ at x-value $a = 2$.
4. $f(x) = \tan x$ at x-value $a = \pi/4$.

Module 3 Review Exercises

1-3: For each function calculate $f'(x)$ using the definition of the derivative.

Answers: Page 51

1. $f(x) = x^3 + 2$
2. $f(x) = \dfrac{4x-3}{x+2}$
3. $f(x) = \sqrt{2x+1}$

4-10: Differentiate the functions.

4. $y = 3x^4 + \sqrt[3]{2x} - \dfrac{5}{\sqrt{x}} + \pi$
5. $g(x) = \left(\sqrt{2x} - 4x + 3\right)(3x + \sin x)$
6. $h(y) = \dfrac{\sqrt{y+5}}{3y+2}$
7. $f(\theta) = \cos^2\theta + 4\cos\left(\theta^2\right)$
8. $g(x) = \sec\left(x^3+4\right)\cos(2x)$
9. $f(x) = \sqrt[5]{\dfrac{x^3-4x+10}{4x^2+5}}$
10. $x^4y^3 + 4y^2 = xy + \sin y$

---

11. Find the equation of the tangent line to the curve $y = 3\sin x - 2\cos(3x)$ at the point $P(\frac{\pi}{2}, 3)$.
12. Find the value(s) of x for which the curve $y = \dfrac{x+1}{x^2+3}$ has a horizontal tangent line.
13. Find the value(s) of θ for which the curve $f(\theta) = \cos(2\theta) - 2\cos\theta$ has a horizontal tangent line.
14. The height h and radius r of a circular cone are increasing at the rate of 3 cm/sec. How fast is the volume of the cone increasing when $h = 8$ cm and $r = 3$ cm?
15. A right triangle has a constant height of 30 cm. If the base of the right triangle is increasing at a rate of 6 cm/sec, how fast is the angle between the hypotenuse and the base changing when the base is 30 cm?
16. If the area of an equilateral triangle is increasing at a rate of 5 cm^2/sec, find the rate at which the length of a side is changing when the area of the triangle is 100 cm^3.

Module 4

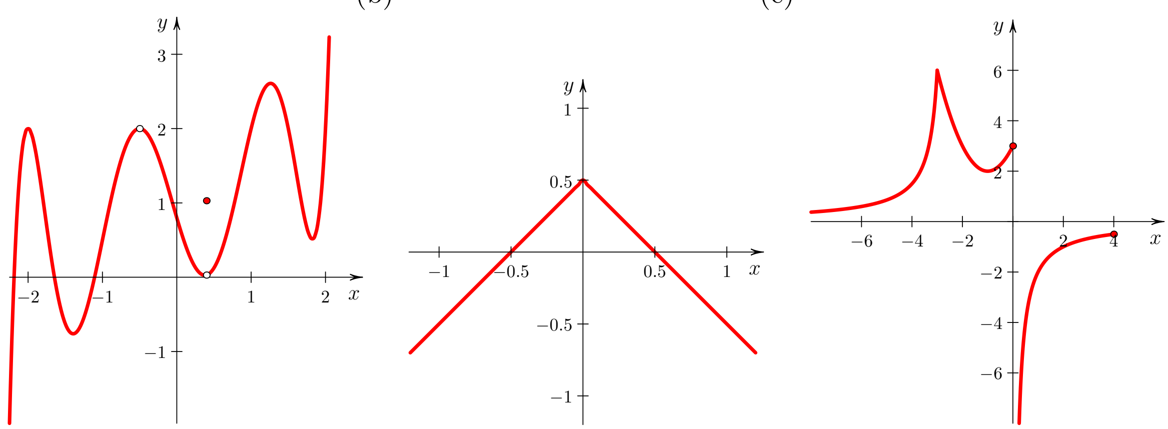
Derivative Applications

4.1 Relative and Absolute Extreme Values

1. Identify the relative maxima, relative minima, absolute maxima, and absolute minima on each of the following graphs.

Answers: Page 52

(a) (b) (c)



2-10: Find the critical numbers of the given function.

2. $f(x) = x^3 - 9x^2 + 24x - 15$
3. $h(x) = |x| + 1$
4. $f(s) = \dfrac{s}{s^2+6}$ on the interval $[0, 10]$.
5. $f(x) = x^3 + 5x^2 + 3x + 1$
6. $g(t) = \dfrac{1}{4}t^4 + 2t^2 - 5t + 6$
7. $H(x) = \dfrac{x+3}{x-5}$
8. $f(t) = \sqrt{t^2 - 4}$
9. $g(x) = \sqrt[3]{x^2 - 5}$
10. $F(\theta) = 2\sin(\theta) - \theta$

11-15: Find the absolute maximum and absolute minimum values and their locations for the given function on the closed interval.

11. $f(x) = x^4 - x^2 + 1$ on $[-2, 2]$
12. $f(x) = x^3 + 5x^2 + 3x + 1$ on $[-1, 0]$.
13. $g(t) = \sqrt{t}\,(t-2)$ on $[0, 1]$.
14. $H(x) = x^{\frac{1}{3}} - 3$ on $[-1, 8]$.
15. $f(x) = \sin x \cos x$ on $[0, 2\pi]$

4.2 Rolle's Theorem and the Mean Value Theorem

Answers: Page 53

1. Using Rolle's Theorem, show that the graph of $f(x) = x^3 - 2x^2 - 7x - 2$ has a horizontal tangent line at a point with x-coordinate between -1 and 4. Next find the value $x = c$ guaranteed by the theorem at which this occurs.

2. By Rolle's Theorem it follows that if f is a function defined on $[0, 1]$ with the following properties:

   (a) f is continuous on $[0, 1]$
   (b) f is differentiable on $(0, 1)$
   (c) $f(0) = f(1)$

   then there exists a least one value c in $(0, 1)$ with $f'(c) = 0$. Show that each condition is required for the conclusion to follow by giving a counterexample in the case that (a), (b), or (c) is not required.

3. Suppose that f is continuous on $[-3, 4]$, differentiable on $(-3, 4)$, and that $f(-3) = 5$ and $f(4) = -2$. Show there is a c in $(-3, 4)$ with $f'(c) = -1$.

4. Verify the Mean Value Theorem for the function $f(x) = x^3 + 2x - 2$ on the interval $[-1, 2]$.

4.3 First Derivative Test

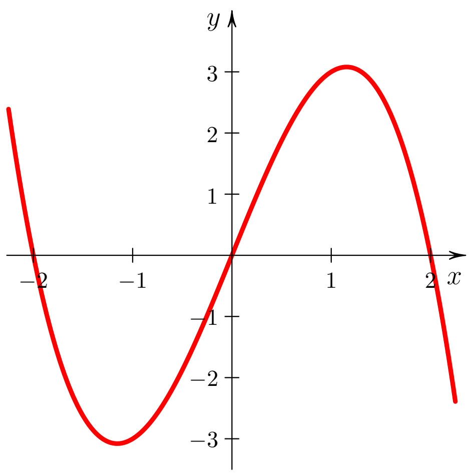
1-5: Find the open intervals upon which the following functions are increasing or decreasing. Also find any relative minima and maxima and their locations.

Answers: Page 53

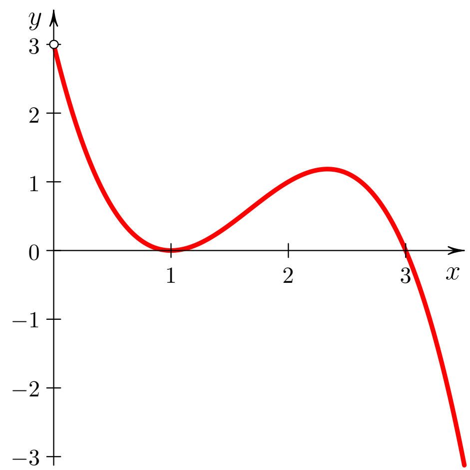
1. $f(x) = x^2 + 2x + 1$
2. $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$
3. $f(x) = 4\cos x - 2x$ on $[0, 2\pi]$
4. $f(x) = |x|$
5. $f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

6-7: Each graph below is a graph of the derivative f' of a function f. In each case use the graph of f' to sketch a possible graph of f.

6.



7.



4.4 Inflection Points

1-2: Show that the following functions have no inflection points:

Answers: Page 54

1. $f(x) = 2x^4$
2. $f(x) = \dfrac{1}{x}$

---

3. Find the relative extrema (and their locations), the intervals of concavity, and the inflection points of the function $f(x) = x^5 - 15x^3 + 1$.

4.5 Second Derivative Test

Answers: Page 54

1-2: Determine the relative maxima and minima of the following functions and their locations. Use the Second Derivative Test.

1. $f(x) = x^3 - 12x + 1$
2. $f(x) = \cos(2x) - 4\sin(x)$ on the interval $(-\pi, \pi)$

---

3. Can you use the Second Derivative Test to categorize the critical number $x = 0$ of the function $f(x) = \sin^4 x$? Explain why or why not.

4.6 Curve Sketching I

Answers: Page 55

1-3: Find the domain, intercepts, intervals of increase and decrease, relative maxima and minima, intervals of concave upward and downward, and inflection points of the given functions and then sketch their graphs.

1. $f(x) = x^3 - 6x^2$
2. $g(t) = t^{\frac{3}{2}} - 3t^{\frac{1}{2}}$
3. $F(x) = \sqrt{x^2 + 9}$

4.7 Limits at Infinity and Horizontal Asymptotes

Answers: Page 56

1-13: Determine the following limits. For any limit that does not exist, identify if it has an infinite trend (∞ or $-\infty$).

1. $\displaystyle\lim_{x\to-\infty} \frac{18x^2 - 3x}{3x^5 - 3x^2 + 2}$
2. $\displaystyle\lim_{x\to\infty} \frac{x^2}{\sqrt{x^3 + 2}}$
3. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2 + 2x}}{6x + 3}$
4. $\displaystyle\lim_{x\to\infty} \left[\sec\left(\frac{1}{x}\right) + 1\right]$
5. $\displaystyle\lim_{x\to-\infty} \sqrt{x^2 + 3x + 5} + x$
6. $\displaystyle\lim_{x\to\infty} \frac{x^2 + 5x + 4}{3x^2 + 2}$
7. $\displaystyle\lim_{x\to\infty} \frac{4x^5 - 3x^2 + 6}{x^4 + 7}$
8. $\displaystyle\lim_{x\to\infty} \frac{3x^4 + 6x - 7}{x^5 + 10}$
9. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2 + 10}}{x + 3}$
10. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2 + 3}}{x - 2}$
11. $\displaystyle\lim_{x\to-\infty} \frac{5x + \sqrt{x^2 + 1}}{x + 5}$
12. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2 + 4x + 1} - x\right)$
13. $\displaystyle\lim_{x\to-\infty} \left(x + \sqrt{x^2 - 6x + 5}\right)$

14-21: Find the horizontal asymptotes of the following functions.

14. $f(x) = \dfrac{3x+3}{2x-4}$

15. $f(x) = x^3 + 5x + 2$

16. $g(t) = \dfrac{\sqrt{t^2+3}}{t-2}$

17. $f(x) = \dfrac{x^2-2x+1}{2x^2-2x-12}$

18. $f(x) = \dfrac{\cos x}{x}$

19. $y = \dfrac{5x^2-3x+1}{x^2-16}$

20. $f(x) = \dfrac{x^3+1}{x^3+x^2}$

21. $F(x) = \dfrac{x}{\sqrt{4x^2+1}}$

4.8 Slant Asymptotes

1-3: Find any slant asymptotes of the graphs of the following functions.

Answers: Page 56

1. $f(x) = \dfrac{3x^2-4x}{x+2}$

2. $g(x) = \dfrac{x^3+2x+1}{x^4+5x-7}$

3. $y = \dfrac{x^3-2x^2+1}{x^2+2}$

4.9 Curve Sketching II

1-3: Apply calculus techniques to identify all important features of the graph of each function and then sketch it.

Answers: Page 57

1. $f(x) = x^3 - 3x - 2$

2. $y = \dfrac{3x^2}{x^2-1}$

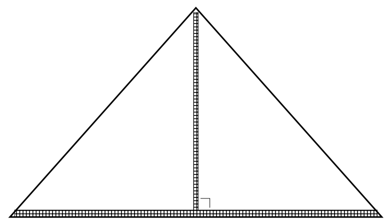
3. $f(x) = \dfrac{x^2}{x^2+2x+1}$

4.10 Optimization Problems

1-11: Solve the following optimization problems following the steps outlined in the text.

Answers: Page 58

1. The entrance to a tent is in the shape of an isosceles triangle as shown below . Zippers run vertically along the middle of the triangle and horizontally along the bottom of it. If the designers of the tent want to have a total zipper length of 5 metres, find the dimensions of the tent that will maximize the area of the entrance. Also find this maximum area.

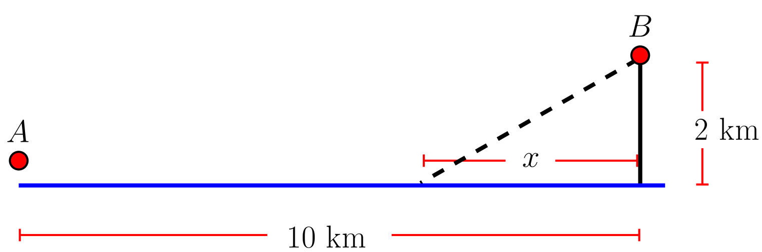


2. Find the point on the line $y = 2x + 2$ closest to the point $(3, 2)$ by
   (a) Using optimization.
   (b) Finding the intersection of the original line and a line perpendicular to it that goes through $(3, 2)$.

3. The product of two positive numbers is 50. Find the two numbers so that the sum of the first number and two times the second number is as small as possible.

4. A metal cylindrical can is to be constructed to hold 10 cm^3 of liquid. What is the height and the radius of the can that minimize the amount of material needed?

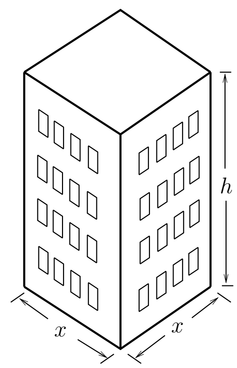
5. A pair of campers wish to travel from their campsite along the river (location A) to visit friends 10 km downstream staying in a cabin that is 2 km from the river (location B) as shown in the following diagram:



   (a) If the pair can travel at 8 km/h in the river downstream by canoe and 1 km/h carrying their canoe by land, at what distance x (see diagram) should they depart from the river to minimize the total time t it takes for their trip?
   (b) On the way back from the cabin they can only travel at 4 km/h in their canoe because they are travelling upstream. What distance x will minimize their travel time in this direction?
   (c) Using the symbolic constants a for the downstream distance, b for the perpendicular land distance, w for the water speed and v for the land speed, find a general expression for optimal distance x. Verify your results for parts (a) and (b) of this problem by substituting the appropriate constant values.

(d) Does your general result from part (c) depend at all on the downstream distance a? Discuss.

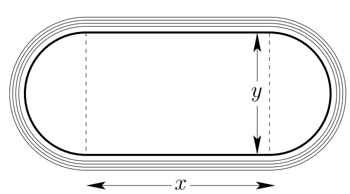
6. A construction company desires to build an apartment building in the shape of a rectangular parallelpiped (shown) with fixed volume of 32000 m^3. The building is to have a square base. In order to minimise heat loss, the total above ground surface area (the area of the four sides and the roof) is to be minimised. Find the optimal dimensions (base length and height) of the building.



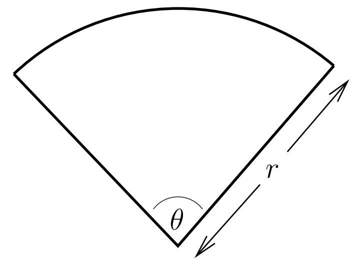
7. A rectangular field is to be enclosed and then divided into three equal parts using 32 metres of fencing. What are the dimensions of the field that maximize the total area?

8. Two power transmission lines travel in a parallel direction (north-south) 10 km apart. Each produces electromagnetic interference (EMI) with the one to the west producing twice the EMI of that of the one to the east due to the greater current the former carries. An amateur radio astronomer wishes to set up his telescope between the two power lines in such a way that the total electromagnetic interference at the location of the telescope is minimized. If the intensity of the interference from each line falls off as 1/distance, how far should the telescope be positioned from the stronger transmission line?

9. An athletic field consists of a rectangular region with a semicircular region at each end. The perimeter of the entire athletic field has to be 400 metres. Find the dimensions that maximize the area of the rectangular region.



10. An antenna is to be created by bending a wire of length 16 cm into the shape of a sector of a circle as shown. In order to maximize the electric flux through the wire, the area of the sector is to be maximized. Find the dimensions r and θ that will maximize the area. Also find the maximum area.



11. A *tipi* is a conical tent, typically having no floor, constructed by placing skins, cloth, or canvas over a frame of poles. In addition to its practical use as a shelter, it is an important symbol of the First Nations of the North American plains.

    (a) Assuming a large number of poles, a tipi is approximately a right circular cone with base radius r and height h. If a tipi has a fixed lateral surface area of A, find the radius and height in terms of A that maximize its volume. What is the ratio h/r for the optimal tipi?

    (b) An actual tipi having n poles will have a conical base that is a regular n-sided polygon rather than a circle. Repeat part (a) by finding the apothem H of the regular polygon base and height h of the tipi that maximize its volume for fixed lateral surface area A. Find the ratio h/H for the optimal tipi. Show that in the limit $n \to \infty$ you recover the solution from part (a).

Module 4 Review Exercises

1-3: Find the critical numbers of the given functions.

Answers: Page 59

1. $f(x) = \dfrac{x+2}{x^2-3}$
2. $g(t) = \sqrt{t^2 - 3t}$
3. $F(\theta) = \cos(2\theta) + 2\sin\theta$

4-5: Find the absolute maximum and absolute minimum values and their locations for the given function on the closed interval.

4. $f(x) = \dfrac{x}{x^2+16}$ on the closed interval $[-1,\ 1]$.
5. $g(t) = t\sqrt{8-t^2}$ on the closed interval $[0, 1]$.

6-7: Find the domain, intercepts, asymptotes, relative maxima and minima, intervals of increase and decrease, intervals of concave upward and downward, and inflections points. Then sketch the graph of the given functions.

6. $f(x) = 8x^{\frac{1}{3}} + x^{\frac{4}{3}}$
7. $g(x) = \dfrac{x^2}{x-2}$

8-11: Evaluate the given limits.

8. $\displaystyle\lim_{x\to\infty} \frac{3x^2-4x-5}{2x^5+3}$
9. $\displaystyle\lim_{x\to-\infty} \frac{5x^4-3x+1}{x^4+7}$
10. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{5x^2-4}}{2x+1}$
11. $\displaystyle\lim_{x\to-\infty} \left(2x + \sqrt{x^2+4x+2}\right)$

12-13: Find the horizontal and vertical asymptotes of the given functions.

12. $y = \dfrac{2x^2+7x+3}{x^2+x-6}$
13. $g(t) = \dfrac{\sqrt{9+4t^2}}{2t+3}$

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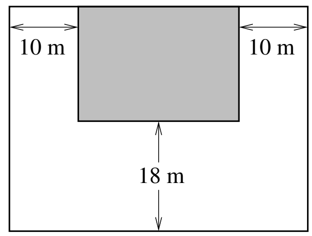
14. A store with a rectangular floorplan is to sit in the middle of one side of a larger rectangular lot with parking on three sides as shown.



If the narrow strips of parking on the side are to be 10 m wide while the parking in the front is to be 18 m wide, find the optimal dimensions of the store that will minimize the total lot area if the store itself must have an area of 1000 m^2. What is the total lot area in this case?

15. A farmer wants to enclose a rectangular garden on one side by a brick wall costing \$20/m and on the other three sides by a metal fence costing \$5/m. Find the dimensions of the garden that minimize the cost if the area of the garden is 250 m^2.

16. A window shaped like a Roman arch consists of a rectangle surmounted by a semicircle. Find the dimensions of the window that will allow the maximum amount of light if the perimeter of the window is 10 m.

Module 5

Integration

5.1 Antiderivatives

1. Why are $F_1(x) = \frac{1}{4}x^4$ and $F_2(x) = \frac{1}{4}(x^4 + 2)$ both antiderivatives of $f(x) = x^3$?

Answers: Page 60

2-6: Find the antiderivative of the given functions.

2. $f(x) = 3x^2 - 5x + 6$
3. $f(x) = \dfrac{x^3 + 4}{x^2}$
4. $g(t) = \sqrt{t} + \dfrac{2}{\sqrt{t}}$
5. $h(x) = \sqrt[3]{x^2} - 4x^6 + \pi$
6. $f(\theta) = 2\cos\theta - \sin\theta + \sec^2\theta$

7-9: Find the function(s) f satisfying the following.

7. $f''(x) = 2x^3 - 10x + 3$
8. $f''(t) = \sqrt{t} + 6t, \quad f(1) = 1, \quad f'(1) = 2$
9. $f''(\theta) = 3\sin\theta + \cos\theta + 5, \quad f(0) = 3, \quad f'(0) = -1$

---

10. Suppose f is a function with $f'''(x) = 0$ for all x. Show that f has no points of inflection.

5.2 Series

Answers: Page 61

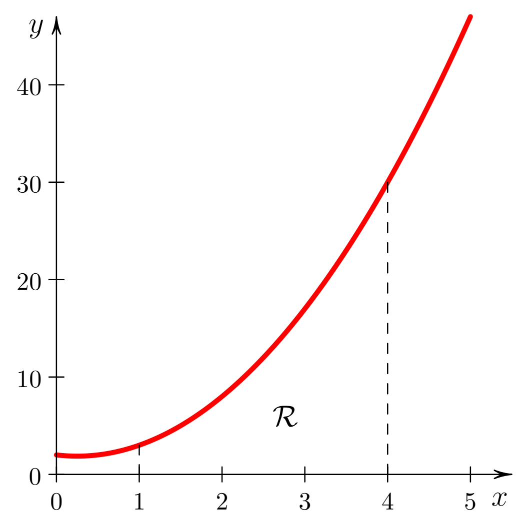
1-4: Evaluate the sums of the following series. (Any value that is not an index being summed over should be treated as a postive integer constant.)

1. $\displaystyle\sum_{i=2}^{5} \frac{i+2}{i-1}$
2. $\displaystyle\sum_{k=1}^{4} 6k$
3. $\displaystyle\sum_{i=1}^{n} \frac{i^2+1}{n^3}$
4. $\displaystyle\sum_{i=1}^{n} i(i-3)$

5.3 Area Under a Curve

Answers: Page 61

1. Let A be the area of the region $\mathcal{R}$ bounded by the x-axis, the lines $x = 1$ and $x = 4$, and the curve $f(x) = 2x^2 - x + 2$ shown below:



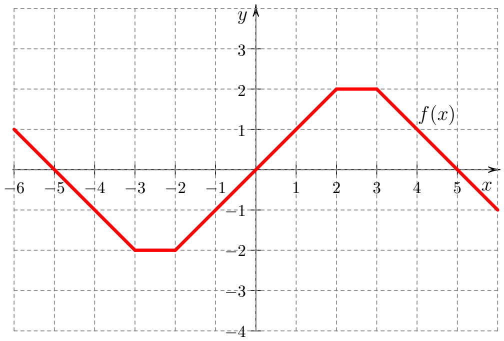
   (a) Write a formula for the n^{th} sum S_n approximating A using right endpoints of the approximating rectangles.

   (b) Use $A = \lim_{n\to\infty} S_n$ and your answer from (a) to calculate A.

5.4 The Definite Integral

1. Consider the following graphically defined function $f(x)$:

Answers: Page 61



   (a) Using the interpretation of the definite integral in terms of net signed area between the function and the x-axis, find

      i. $\int_{-5}^{0} f(x)\,dx$

      ii. $\int_{-2}^{5} f(x)\,dx$

   (b) If we define the function $g(t) = \int_{-6}^{t} f(x)\,dx$, on what intervals is

      i. $g(t)$ increasing?

      ii. $g(t)$ decreasing?

2-5: Use the interpretation of the definite integral as the net signed area to find:

2. $\int_{0}^{2} 3x\,dx$ 3. $\int_{-1}^{4} 6\,dx$ 4. $\int_{0}^{3} (2x-4)\,dx$ 5. $\int_{-4}^{3} (-x)\,dx$

---

6. Find $\int_{-r}^{0} \sqrt{r^2 - x^2}\,dx$ by interpreting the integral as an area. Here $r > 0$ is a positive constant.

7. Simplify the following to a single definite integral using the properties of the definite integral.

$$\int_{-1}^{7} f(x)\,dx + \int_{3}^{-1} f(x)\,dx + \int_{7}^{9} f(x)\,dx$$

8. Use properties of the definite integral to verify the following without evaluating the integral.

$$2 \leq \int_{-1}^{1} (2 + \sin x)\,dx \leq 6$$

9-10: Use Riemann sums with right endpoint evaluation to evaluate the following definite integrals.

9. $\displaystyle\int_0^3 \left(x^3+1\right) dx$

10. $\displaystyle\int_0^b x^2\, dx$ where $b>0$ is constant

5.5 Fundamental Theorem of Calculus (Derivative Form)

Answers: Page 62

1-4: Compute the derivatives of the following functions using the Fundamental Theorem of Calculus.

1. $F(x)=\displaystyle\int_0^x \sqrt{t^3+2t+1}\, dt$

2. $h(x)=\displaystyle\int_0^{x^4} \sqrt{t^3+2t+1}\, dt$

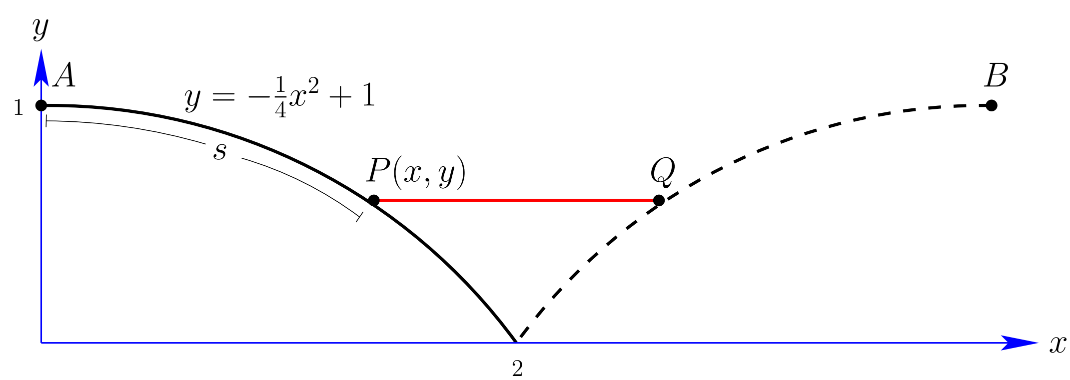
3. $g(x)=\displaystyle\int_x^1 \left[\cos\left(t^3\right)\right] dt$

4. $H(x)=\displaystyle\int_{2x}^{3x} \sqrt[3]{t^3+1}\, dt$

---

5. The *error function*, $\operatorname{erf}(x)$, is defined by $\operatorname{erf}(x)=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_{-\infty}^x e^{-t^2}\, dt$ where e^u is the natural exponential function. If $f(x)=\operatorname{erf}\left(x^3\right)$ find $f'(x)$.

6. The left-side of a symmetrical glacial river valley has an approximate parabolic shape described by the curve $y=-\frac{1}{4}x^2+1$ (in km) as shown.



Using calculus techniques the arc length s from point A at the top of the valley to the point $P(x,y)$ can be shown to equal

$$s=\int_0^x \sqrt{1+\frac{1}{4}t^2}\, dt\ .$$

Engineers wish to build a road connecting points A and B with a bridge spanning the valley at the point P to the corresponding point Q on the opposite side of the valley. If the cost to build the bridge is 25% more per kilometre than the cost of building the road (i.e. it is 5/4 times as much per km), at what point $P(x,y)$ should they start the bridge to minimize the total cost? (Hint: Due to the symmetry of the situation just minimize the cost to build from point A to the middle of the bridge.)

5.6 Fundamental Theorem of Calculus (Antiderivative Form)

1-7: Compute the following definite integrals using the Fundamental Theorem of Calculus.

Answers: Page 62

1. $\displaystyle\int_{1}^{3} \left(x^2 + 3\right) dx$
2. $\displaystyle\int_{4}^{1} \sqrt{x}\, dx$
3. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 \theta \, d\theta$
4. $\displaystyle\int_{-2}^{-1} \frac{4}{x^4}\, dx$
5. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin x + x)\; dx$
6. $\displaystyle\int_{-3}^{2} |x|\, dx$
7. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$

5.7 Indefinite Integrals

1. Explain why we use the indefinite integral symbol, $\int f(x)\, dx$, to represent the general form of the antiderivative of the function $f(x)$.

Answers: Page 63

2-5: Evaluate the following indefinite integrals.

2. $\displaystyle\int \left(x^3 - 3x^4 - 6\right)\; dx$
3. $\displaystyle\int \frac{2+x}{\sqrt{x}}\, dx$
4. $\displaystyle\int \csc\theta \cot\theta \, d\theta$
5. $\displaystyle\int \left(\tan^2 x + 1\right)\; dx$

---

6. Find the general form of the function $y = f(x)$ such that the equation $y' = x^2 + 9$ is satisfied.

5.8 The Substitution Rule

1-6: Evaluate the following indefinite integrals using the Substitution Rule.

Answers: Page 63

1. $\displaystyle\int \frac{x^2 + 2x}{\left(x^3 + 3x^2 + 4\right)^5}\, dx$
2. $\displaystyle\int (5x+1)\sqrt{5x^2 + 2x}\, dx$
3. $\displaystyle\int \left(\frac{\cos\sqrt{t}}{\sqrt{t}} + t^3\right)\; dt$
4. $\displaystyle\int \cos(\theta)\sqrt{3 - \sin\theta}\, d\theta$
5. $\displaystyle\int x\sqrt{4x+1}\, dx$
6. $\displaystyle\int \sec^2\left(2x - \frac{\pi}{3}\right) dx$

7-12: Evaluate the following definite integrals using the Substitution Rule.

7. $\displaystyle\int_0^2 x^3\sqrt{x^4+9}\,dx$

8. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4\theta\sec^2\theta\,d\theta$

9. $\displaystyle\int_0^1 \left[1+x+(1-x)^5\right]\,dx$

10. $\displaystyle\int_1^3 \frac{x}{\left(2x^2+1\right)^2}\,dx$

11. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin t}{\cos^{\frac{2}{3}} t}\,dt$

12. $\displaystyle\int_0^{\frac{T}{3}} \cos\left(\frac{2\pi t}{T}\right)\,dt$ ($T>0$ is constant)

5.9 Integration Using All Methods

1-11: Evaluate the given integrals using any method.

Answers: Page 64

1. $\displaystyle\int \left(x^3+\sqrt{x}-\frac{1}{x^2}+5\right)\,dx$

2. $\displaystyle\int \left(x^3+2\right)^2\,dx$

3. $\displaystyle\int \frac{\left(2x+\sqrt{x}\right)^2}{\sqrt{x}}\,dx$

4. $\displaystyle\int \left(\cos\theta+\sec^2\theta\right)\,d\theta$

5. $\displaystyle\int 3x^2\left(x^3+4\right)^5\,dx$

6. $\displaystyle\int \cos\theta\left(\sin\theta+3\right)^{10}\,d\theta$

7. $\displaystyle\int \frac{\left(\sqrt{t}+7\right)^{\frac{4}{3}}}{\sqrt{t}}\,dt$

8. $\displaystyle\int_{-1}^2 (2x+3)^4\,dx$

9. $\displaystyle\int_0^1 \frac{x^2}{\left(x^3+2\right)^3}\,dx$

10. $\displaystyle\int_0^{\frac{\pi}{4}} \sin(2\theta)\,d\theta$

11. $\displaystyle\int_{-\frac{3}{2}}^{\frac{3}{2}} \sin(\tan x)\,dx$

5.10 Area Between Curves

1-7: Find the area of the region bounded by the given curves.

Answers: Page 65

1. $y=x^2-3x+8$ and $y=4x-x^2$ over the closed interval $[-1,2]$.
2. $y=x^2-5x-1$ and $y=x-6$ over the closed interval $[1,6]$.
3. $y=x^2+6,\ \ y=2x^2+2$
4. $x=2y^2,\ \ x=y^2+4$
5. $y=x^2-2x,\ \ y=x-2$
6. $y=0,\ \ y=2x,\ \ x+y=3$
7. $y=\dfrac{8}{x^2}$, $y=x$, $y=4x^2+4x$ and lying in the first quadrant ($x>0$ and $y>0$).

5.11 Distances and Net Change

Answers: Page 65

1. A particle oscillates in a straight line with velocity $v(t) = \sin(\pi t)$ centimetres per second. Compute the particle's displacement over the following time intervals.

   (a) $t = 0$ to $t = 1$ seconds.

   (b) $t = 1$ to $t = 2$ seconds.

   (c) $t = 0$ to $t = 2$ seconds.

2. The flow rate at a particular location for a large river over the month of May was approximately $f(t) = -\frac{1}{3}(t-15)^2 + 100$ in gigalitres per day. Here the time t is measured in days from the beginning of the month. How much water flowed past that location between the times $t = 10$ and $t = 20$ days?

Module 5 Review Exercises

Answers: Page 65

1-4: Find the antiderivative of the given functions.

1. $f(x) = \sqrt[5]{x^3} + \dfrac{4}{\sqrt{x}} + x^3 + 10$

2. $g(x) = \dfrac{\sqrt{x^3} + 5x + 1}{2x^3}$

3. $f(\theta) = 3\sin\theta + 5\cos\theta + \theta^3 + 1$

4. $g(\theta) = 2\tan\theta\sec\theta - 2\cos\theta + \dfrac{1}{\cos^2\theta}$

5-7: Find function f satisfying the given conditions.

5. $f''(x) = \sqrt{x} + x^2 - 6$

6. $f''(t) = 20\sqrt[3]{x^2} - 3x - 5,\ f(1) = 1,\ f'(1) = -1$

7. $f''(\theta) = 5\sin\theta - 4\cos\theta + 10,\ f(0) = -13,\ f'(0) = 2$

8-13: Evaluate the given integrals.

8. $\displaystyle\int x^4\left(x^5+3\right)^8\,dx$

9. $\displaystyle\int \sec^2(2\theta)\left[\tan(2\theta)+1\right]^5\,d\theta$

10. $\displaystyle\int \frac{\left(\sqrt[5]{t^2}-4\right)^{\frac{1}{3}}}{\sqrt[5]{t^3}}\,dt$

11. $\displaystyle\int_{-1}^{1} x(x-1)^6\,dx$

12. $\displaystyle\int_0^{\pi/4} \cos^4(3x)\sin(3x)\,dx$

13. $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2(3\theta)\,d\theta$

14-16: Find the area of the region bounded by the given curves.

14. $y = x^2 + 1$ and $y = 5$.

15. $y = x$, $y = 4x$, and $x + y = 3$.

16. $x = 10 - y^2$ and $x = 2 + y^2$.

17-18: Solve the following differential equations and initial value problems.

17. $y' = \dfrac{3 + x^5}{x^2}$

18. $y' \sec x = 2$, $y(0) = 3$

Answers

1.1 Exercises (page 3)

1. $x = 3$
2. $x = 3$ or $x = -\dfrac{1}{2}$. Written as a *solution set* it is $\{3, -1/2\}$
3. $x = \dfrac{-3 \pm \sqrt{-7}}{8}$, so no real solution.
4. $x = 2$
5. $\{-2, 2, 4\}$
6. $x = 1$
7. $\left\{0, \pm\sqrt{\dfrac{3}{2}}\right\}$
8. $\left\{\pm 1, \pm\sqrt{2}\right\}$
9. $\left\{-1, \pm\sqrt{\dfrac{2}{3}}\right\}$
10. $\left\{-1, 0, \dfrac{-3 \pm \sqrt{33}}{4}\right\}$

1.2 Exercises (page 3)

1. $D = \mathbb{R} = (-\infty, \infty)$; x-int=0; y-int=0
2. $D = [6, \infty)$; x-int= 6; No y-int
3. $D = \mathbb{R} = (-\infty, \infty)$; x-int=-2; y-int=4
4. $D = \mathbb{R} = (-\infty, \infty)$; x-int=-5, 0 ,1; y-int=0
5. $D = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$; No x-int; y-int= -1
6. $D = \mathbb{R} - \{-2\} = (-\infty, -2) \cup (-2, \infty)$; No x-int; y-int= $\dfrac{1}{4}$
7. $D = [-2, 2]$; x-int=-2, 2; y-int=2

8. $D = \mathbb{R} = (-\infty, \infty)$; x-int$= -\dfrac{2}{3}, -1$; y-int$= 2$

9. $D = \mathbb{R} = (-\infty, \infty)$; x-int$= 1$; y-int$= -4$

10. $D = \mathbb{R} - \{-7\} = (-\infty, -7) \cup (-7, \infty)$; x-int$= -5$; y-int$= \dfrac{5}{7}$

11. $D = \mathbb{R} - \{-1/2, 3\} = (-\infty, -1/2) \cup (-1/2, 3) \cup (3, \infty)$; No x-int; y-int$= -\dfrac{10}{3}$

12. $D = \{x \in \mathbb{R} | x \leq 4\} = (-\infty, 4]$; x-int$= 4$; y-int$= 2$

13. $D = \left(-\infty, -\sqrt{10}\right] \cup \left[\sqrt{10}, \infty\right)$; x-int$= \pm\sqrt{10}$; No y-int

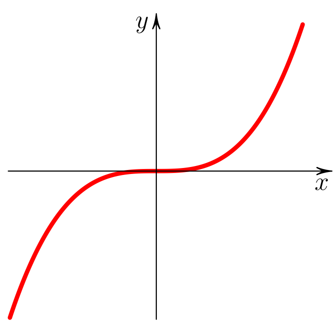
14. $D = (-\infty, -6] \cup (3/2, \infty)$; x-int$= -6$; No y-int

15. $D = \left(-\infty, -\sqrt{10}\right] \cup \left[\sqrt{10}, \infty\right)$; x-int$= \pm\sqrt{10}$; No y-int

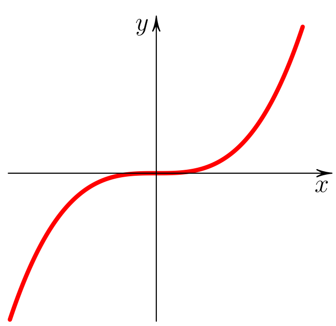
16. $D = (-2, 0) \cup (0, \infty)$; x-int$=2$; No y-int

17. Even	21. Neither	25. Even	29. Odd
18. Odd	22. Odd	26. Odd	30. Neither
19. Neither	23. Even	27. Neither	31. Even
20. Odd	24. Even	28. Even	32. Even

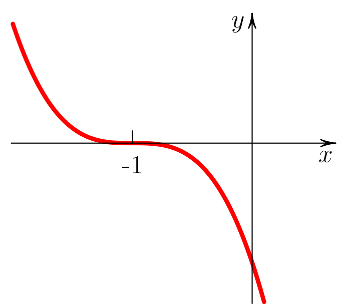
33. (a)



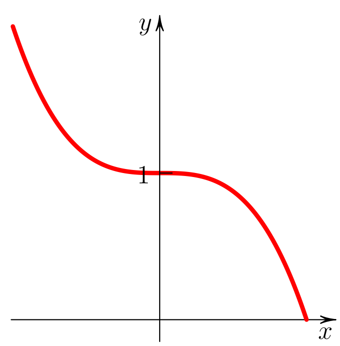
(b)



(c)



(d)



(e) $f(-x) = -f(x)$, and so f is odd.

1.3 Exercises (page 5)

1. (a) $f(x+2) = \dfrac{1}{x+4}$ (b) $f(f(x)) = \dfrac{x+2}{2x+5}$
2. $f \circ g(x) = f(g(x)) = 2\left(x^2+3\right)^{\frac{3}{2}}$ with $D = \mathbb{R}$, $g \circ f(x) = g(f(x)) = \sqrt{4x^6+3}$ with $D = \mathbb{R}$
3. $f \circ g(x) = f(g(x)) = 3(3x-2)^2 + 6(3x-2) + 4 = 27x^2 - 18x + 4$ with $D = \mathbb{R}$, $g \circ f(x) = g(f(x)) = 3(3x^2+6x+4) - 2 = 9x^2 + 18x + 10$ with $D = \mathbb{R}$
4. $f \circ g(z) = f(g(z)) = \sqrt{\left(\dfrac{z}{z+1}\right)^2 + 5} = \sqrt{\dfrac{6z^2+10z+5}{(z+1)^2}}$ with $D = \mathbb{R} - \{-1\}$, $g \circ f(z) = g(f(z)) = \dfrac{\sqrt{z^2+5}}{\sqrt{z^2+5}+1}$ with $D = \mathbb{R}$
5. $f \circ g(x) = f(g(x)) = \dfrac{2\left(x^2+3\right)+5}{(x^2+3)-4} = \dfrac{2x^2+11}{x^2-1}$ with $D = \mathbb{R} - \{-1, 1\}$, $g \circ f(x) = g(f(x)) = \left(\dfrac{2x+5}{x-4}\right)^2 + 3 = \dfrac{7x^2-4x+73}{(x-4)^2}$ with $D = \mathbb{R} - \{4\}$
6. $f(x) = \sqrt{x} - 3$, $g(x) = x^2 + 1$.
7. $\left\{\pm 1, \pm\sqrt{2}\right\}$

Module 1 Review Exercises (page 6)

1. $x = 1/2$, $x = -2$. Written as a solution set: $\{1/2, -2\}$
2. $\{-5, 4\}$
3. $\{-2, -1, 1\}$
4. $\{\pm 1, \pm 2\}$
5. $\{-2, 1, 2\}$
6. $D = \mathbb{R} - \left\{-\dfrac{4}{5}\right\}$, x-int$= \dfrac{3}{2}$, y-int$= -\dfrac{3}{4}$
7. $D = (-\infty, -2] \cup [2, \infty)$, x-int=$2, -2$, y-int does not exist
8. $D = (-\infty, -5) \cup \left[-\dfrac{1}{2}, \infty\right)$, x-int$= -\dfrac{1}{2}$, y-int$= \sqrt{\dfrac{1}{5}}$
9. $D = [-8, 3) \cup (3, \infty)$, x-int$= -8$, y-int$= -\dfrac{\sqrt{8}}{3}$
10. Even
11. Even
12. Odd
13. Odd
14. $f \circ g(x) = f(g(x)) = x^2 + 6$ with $D = \mathbb{R}$, $g \circ f(x) = g(f(x)) = \sqrt[3]{(x^3+6)^2}$ with $D = \mathbb{R}$

15. $f \circ g(t) = f(g(t)) = \dfrac{2t^2+11}{t^2-1}$ with $D = \mathbb{R} - \{1,-1\}$, $g \circ f(t) = g(f(t)) = \dfrac{4t^2+20t+25}{(t-4)^2} + 3$ with $D = \mathbb{R} - \{4\}$

16. $f \circ g(x) = f(g(x)) = \sqrt{\dfrac{2}{x+3}}$ with $D = (-3, \infty)$, $g \circ f(x) = g(f(x)) = \dfrac{\sqrt{x-1}+5}{\sqrt{x-1}+3}$ with $D = [1, \infty)$

2.1 Exercises (page 7)

1. (a) $(-1)^3 + (-1)^2 - 2(-1) + 3 = 5$, $(0)^3 + (0)^2 - 2(0) + 3 = 3$
   (b) $\dfrac{3-5}{0-(-1)} = -2$
   (c) $m(x) = \dfrac{x^3+x^2-2x+3-5}{x-(-1)} = x^2 - 2 \quad (x \neq -1)$
   (d) $m_t = \lim\limits_{x \to -1} m(x) = -1$

2.2 Exercises (page 7)

1. $\dfrac{13}{5}$
2. 0
3. $\dfrac{1}{4}$
4. 0
5. $\dfrac{7}{4}$
6. $-\dfrac{1}{9}$
7. 1
8. 1
9. $\dfrac{1}{4}$
10. $-\dfrac{3}{10}$
11. -6
12. $-\dfrac{1}{25}$
13. -10
14. 12
15. $\dfrac{7}{3}$

2.3 Exercises (page 8)

1. 1
2. 1
3. 0
4. $\dfrac{7}{5}$
5. 0
6. 1
7. $-\dfrac{1}{\pi}$
8. $\dfrac{4}{\pi}$
9. 0
10. 1
11. $\dfrac{1}{2}$

2.4 Exercises (page 8)

1. $\lim\limits_{x \to 0^-} f = 4$, $\lim\limits_{x \to 0^+} f = 4$, $\lim\limits_{x \to 2^-} f = 3$, $\lim\limits_{x \to 2^+} f = 2$, $\lim\limits_{x \to 4^-} f = \infty$ (limit does not exist but approaches infinity), $\lim\limits_{x \to 4^+} f = 2$

2. $\lim\limits_{x \to 0^-} f = -1$, $\lim\limits_{x \to 0^+} f = -1$, $\lim\limits_{x \to 2^-} f = 1$, $\lim\limits_{x \to 2^+} f = -2$, $\lim\limits_{x \to 4^-} f = 2$, $\lim\limits_{x \to 4^+} f = 2$, $\lim\limits_{x \to 0} f = -1$, $\lim\limits_{x \to 2} f$ does not exist, $\lim\limits_{x \to 4} f = 2$

3. $\lim\limits_{x \to 2^-} f(x) = \dfrac{2}{3}$, $\lim\limits_{x \to 2^+} f(x) = 5$, $\lim\limits_{x \to 2} f(x)$ does not exist as the left and right-handed limits are not equal at $x = 2$.

4. $c = \dfrac{2}{7}$

2.5 Exercises (page 9)

1. ∞
2. $-\infty$
3. $\frac{6}{7}$
4. ∞
5. Vertical asymptote: $x = 2$
6. No vertical asymptotes
7. Vertical asymptote: $t = 2$
8. Vertical asymptotes: $x = -2$, $x = 3$
9. Vertical asymptote: $x = 0$
10. Vertical asymptotes: $x = -4$, $x = 4$
11. Vertical asymptote: $x = 0$
12. No vertical asymptotes

2.6 Exercises (page 9)

1. "f is continuous at $x = a$" if (a) a is in the domain of f (b) $\lim_{x\to a} f(x)$ exists (c) $\lim_{x\to a} f(x) = f(a)$.
2. Continuous
3. Continuous
4. Removable Discontinuity
5. Continuous
6. Infinite Discontinuity
7. Jump Discontinuity
8. (a) $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0^+} f(x) = 4$. Therefore $\lim_{x\to 0} f(x)$ does not exist.
   (b) $c = \pm 1$
9. $\mathbb{R} - \{-1, 1\}$. Note the limit actually exists at $x = -1$ but the function is not defined there.
10. Let $f(x) = x^3 + 2x^2$. Then notice that $f(1) = 3$ and $f(3) = 33$. Then since $f(1) < 10 < f(3)$ and f is continuous, there exists a $c \in (1, 3)$ such that $f(c) = 10$ by the IVT. Since $f(c) = c^3 + 2c^2$ the result follows.
11. $f(x) = x^2 + \cos x - 2$ is continuous on $[0, 2]$, $f(0) = -1 < 0$, $f(2) \approx 1.58 > 0$, so by the Intermediate Value Theorem there is a c in $(0, 2)$ with $f(c) = 0$. i.e. $c^2 + \cos c - 2 = 0$ and c is therefore a solution to the equation.

Module 2 Review Exercises (page 11)

1. 1
2. $\frac{1}{24}$
3. $-\frac{3}{4}$
4. $-\frac{3}{32}$
5. $-\frac{1}{4}$
6. $\frac{4}{5}$
7. $\frac{3}{4}$
8. 4
9. 0
10. Continuous
11. Not continuous
12. Not continuous

3.1 Exercises (page 13)

1. (a) 8 (b) $m \approx 13$ (c) Point-slope form: $y = 13(x-2)+12$, Slope-intercept form: $y = 13x - 14$
2. (a) 1 cm (b) 17 cm (c) 8 cm/s (d) $v \approx 20$ cm/s
3. (a) $\frac{\Delta V}{\Delta P} \approx -7.47$ L/atm (b) -2.49 L/atm

3.2 Exercises (page 14)

1. $f'(2) = \lim_{h\to 0} \frac{\frac{1}{(2+h)+1} - \frac{1}{2+1}}{h} = -\frac{1}{9}$
2. $g'(4) = \lim_{h\to 0} \frac{\sqrt{2(4+h)} - \sqrt{2(4)}}{h} = \frac{1}{2\sqrt{2}}$
3. $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 + 3 - (x^2+3)}{h} = 2x$
4. $f'(x) = \lim_{h\to 0} \frac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} = -\frac{1}{3x^2}$
5. $f'(x) = \lim_{h\to 0} \frac{(x+2+h)^2 - (x+2)^2}{h} = 2x+4$
6. $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} = \frac{1}{2\sqrt{x+2}}$
7. $f'(x) = \lim_{h\to 0} \frac{\frac{3(x+h)+2}{x+h+1} - \frac{3x+2}{x+1}}{h} = \frac{1}{(x+1)^2}$
8. $f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = -\frac{1}{2x\sqrt{x}} = -\frac{1}{2}x^{-\frac{3}{2}}$
9. $\lim_{h\to 0^-} \frac{\sqrt{(2+h-2)^2}}{h} = \lim_{h\to 0^-} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

   $\lim_{h\to 0^+} \frac{\sqrt{(2+h-2)^2}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$

3.3 Exercises (page 14)

1. (a) Differentiable
   (b) Differentiable
   (c) Not Differentiable. Discontinuous at $x = 0$.
   (d) Not differentiable. At $x = 0$ the left and right hand limits for the derivative are not equal:
   $$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \neq \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h},$$
   therefore the limit itself (the derivative) does not exist. Geometrically no tangent line may be drawn at the point so there can be no derivative as that is the tangent slope.

3.4 Exercises (page 15)

1. $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} - 12x^{11} = \dfrac{1}{2\sqrt{x}} - 12x^{11}$
2. $g'(x) = -\dfrac{5}{2}x^{-\frac{7}{2}} = \dfrac{-5}{2\sqrt{x^7}}$
3. $\dfrac{dy}{dx} = \dfrac{12}{5}x^2$
4. $f'(u) = -4u^{-5} + 4u^3$; $f'(1) = 0$
5. $f'(x) = 6x^5 + 8x^3 + 2x$
6. $f'(x) = \sqrt{3} \cdot \dfrac{1}{2}x^{-\frac{1}{2}} + \dfrac{1}{\sqrt[5]{3}} \cdot \dfrac{1}{5}x^{-\frac{4}{5}} = \dfrac{\sqrt{3}}{2\sqrt{x}} + \dfrac{1}{5\sqrt[5]{3x^4}}$
7. $f'(x) = 2a\sin(\pi/15)x^{2a-1}$
8. $\dfrac{ds}{dt} = -gt + v_0$
9. 1(b): $\left.\dfrac{dy}{dx}\right|_{x=2} = \left[3x^2 + 1\right|_{x=2} = 13$

   2(d): $v(2) = \left.\dfrac{ds}{dt}\right|_{t=2} = \left[3t^2 + 4t\right|_{t=2} = 20$ cm/s

   3(b): $\left.\dfrac{dV}{dP}\right|_{P=3} = -(22.4) \cdot P^{-2}\Big|_{P=3} = -2.49$ L/atm
10. $y = \dfrac{3}{2}x + \dfrac{1}{2}$
11. $x = 0,\ \dfrac{4}{3}$

3.5 Exercises (page 15)

1. (a) $C(2000) = 368.6732 \approx 369$ ppm
   (b) $C'(2000) = 1.7948 \approx 1.79$ ppm/year
   (c) $\dfrac{C(2005) - C(2000)}{C(2000)} \times 100 = 2.53\%$.
2. (a) The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Use similar triangles to show that the radius of the surface of the liquid is $r = \frac{2}{5}y$.
   (b) $\left.\dfrac{dV}{dy}\right|_{y=4\text{ m}} = \dfrac{64\pi}{25} \approx 8.04\ \dfrac{\text{m}^3}{\text{m}}$

3.6 Exercises (page 16)

1. $f'(x) = \left(4x^3 - 6x\right)\left(x^{\frac{1}{3}} - x\right) + \left(x^4 - 3x^2 + 2\right)\left(\frac{1}{3}x^{-\frac{2}{3}} - 1\right)$
2. $f'(x) = \left(3x^2 + \pi\right)\left(2 + x^{-3}\right) + \left(x^3 + \pi x + 2\right)\left(-3x^{-4}\right)$
3. $\frac{dy}{dx} = (2x)\left(x^3 + 2\right)\left(2x^2 + \sqrt{x}\right) + \left(x^2 - 1\right)\left(3x^2\right)\left(2x^2 + \sqrt{x}\right) + \left(x^2 - 1\right)\left(x^3 + 2\right)\left(4x + \frac{1}{2}x^{-\frac{1}{2}}\right)$
4. $\frac{df}{dx} = -\frac{2}{(x-6)^2}$
5. $f'(\theta) = \frac{(2\theta + 3)\left(\theta^2 - 7\right) - \left(\theta^2 + 3\theta - 4\right)(2\theta)}{\left(\theta^2 - 7\right)^2} = -\frac{3\theta^2 + 6\theta + 21}{(\theta^2 - 7)^2}$
6. $g'(x) = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}$; $g'(4) = \frac{47}{16}$
7. $f'(v) = \frac{[(2)(v+4/v)+(2v+3)(1-4/v^2)](v^2+v)-(2v+3)(v+4/v)(2v+1)}{\left(v^2+v\right)^2}$
8. $h'(x) = 2cx + \frac{3}{2\sqrt{x}}\left(2x^2 + x\right) + \left(3\sqrt{x} + 2\right)(4x + 1)$

3.7 Exercises (page 16)

1. Point-slope form: $y = 8(x-2) + 1$, Slope-intercept form: $y = 8x - 15$
2. Point-slope form: $y = -\frac{2}{5}(x-1) + 2$, Slope-intercept form: $y = -\frac{2}{5}x + \frac{12}{5}$
3. $(-2, -12)$, $(2, 4)$

3.8 Exercises (page 16)

1. $f'(x) = 18x\left(x^2 + 3\right)^8$
2. $\frac{dg}{dx} = -\frac{\frac{1}{2\sqrt{x}} + 1}{\left(x + \sqrt{x}\right)^2} = -\frac{2\sqrt{x} + 1}{2\sqrt{x}\,(x^2 + x) + 4x^2}$
3. $f'(t) = -\frac{7(4t+3)}{2\left(2t^2 + 3t + 4\right)^{\frac{3}{2}}}$
4. $y' = \frac{1}{7}\left(\frac{4x+3}{x^2+x}\right)^{-\frac{8}{7}} \frac{4x^2 + 6x + 3}{\left(x^2 + x\right)^2}$
5. $h'(x) = \frac{5nx^{n-1}}{3\left(5x^n + 4c\right)^{\frac{2}{3}}}$
6. $f'(x) = 5\left[\left(2\,x + \sqrt{x}\right)^4 + 3\,x\right]^4 \left[4\left(2\,x + \sqrt{x}\right)^3\left(2 + \frac{1}{2\sqrt{x}}\right) + 3\right]$

3.9 Exercises (page 16)

1. Use the sine addition identity followed by the fundamental limits involving sine and cosine to get $f'(x) = 4\cos 4x$.

2. $f'(x) = 2x\cos x - x^2\sin x$

3. $f'(t) = \dfrac{3t^2(\sin t + \tan t) - t^3\left(\cos t + \sec^2 t\right)}{(\sin t + \tan t)^2}$

4. $\dfrac{dH}{d\theta} = -\csc\theta\cot^2\theta - \csc^3\theta$; $H'(\pi/3) = -\dfrac{10}{3\sqrt{3}}$

5. $f'(x) = (\cos x - \sin x)(\sec x - \cot x) + (\sin x + \cos x)(\sec x\tan x + \csc^2 x)$

6. $\dfrac{df}{d\theta} = 0$

7. (a) Since $\tan(\pi/4) = 1$, $x = \pi/4$ and $y = \pi/4$ indeed satisfy the equation.
   (b) Point-slope form: $y = (1+\pi/2)(x-\pi/4)+\pi/4$, Slope-intercept form: $y = (1+\pi/2)x - \pi^2/8$

3.10 Exercises (page 17)

1. $f'(x) = 12\left(x^8 - 3x^4 + 2\right)^{11}\left(8x^7 - 12x^3\right)$

2. $g'(x) = \dfrac{1}{2}\left(3x^2+2\right)^{-\frac{1}{2}}(6x) = \dfrac{3x}{\sqrt{3x^2+2}}$; $g'(2) = \frac{6}{\sqrt{14}}$

3. $f'(\theta) = 2\theta\cos(\theta^2)$

4. $h'(\theta) = -2\cot\theta\csc^2\theta$

5. $f'(x) = \sec\left[\left(x^3+3\right)\left(\sqrt{x}+x\right)\right]\tan\left[\left(x^3+3\right)\left(\sqrt{x}+x\right)\right]\left[\left(3x^2\right)\left(\sqrt{x}+x\right) + \left(x^3+3\right)\left(\dfrac{1}{2\sqrt{x}}+1\right)\right]$

6. $y' = \left(-4\sin\sqrt[3]{x}\right)\cdot\left(\dfrac{1}{3}x^{-\frac{2}{3}}\right) = -\dfrac{4\sin\sqrt[3]{x}}{3\sqrt[3]{x^2}}$

7. $f'(x) = (-1)\left(3+\sin^2 x\right)^{-2}(2\sin x\cos x) = -\dfrac{2\sin x\cos x}{\left(3+\sin^2 x\right)^2}$

8. $\dfrac{dy}{dx} = -5(\csc x + 2)^4\csc x\cot x + 2x + 1$

9. $y' = \pi\sec^2\theta + \pi\sec^2(\pi\theta)$

10. $g'(x) = \dfrac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\left(1+\dfrac{1}{2}x^{-\frac{1}{2}}\right)\left(x^4-1\right)^7 + \left(\sqrt{x+\sqrt{x}}\right)(7)\left(x^4-1\right)^6\left(4x^3\right)$
    $= \dfrac{(2\sqrt{x}+1)\left(x^4-1\right)^7}{4\sqrt{x}\sqrt{x+\sqrt{x}}} + 28x^3\left(x^4-1\right)^6\sqrt{x+\sqrt{x}}$

11. $f'(x) = 3\left(\dfrac{x-3}{x+1}\right)^2\cdot\dfrac{4}{(x+1)^2} = 12\dfrac{(x-3)^2}{(x+1)^4}$

12. $\dfrac{dA}{dt} = -\omega\sin(\omega t+\phi)$; $\left.\dfrac{dA}{dt}\right|_{t=0} = -\omega\sin(\phi)$

13. $f'(x) = \big(\cos\big[\cos(x^2+x)\big]\big)\cdot\big(-\sin\big(x^2+x\big)\big)\cdot(2x+1)$

14. (a) $-2 = 5(0)+3(0)-2(1)$

    (b) $y = 11x-2$

15. $\left\{\dfrac{7\pi}{6}+2n\pi \mid n \text{ an integer}\right\}\bigcup\left\{\dfrac{11\pi}{6}+2n\pi \mid n \text{ an integer}\right\}\bigcup\left\{\dfrac{\pi}{2}+n\pi \mid n \text{ an integer}\right\}$

3.11 Exercises (page 17)

1. $y' = \dfrac{3-2x}{2y}$

2. $y' = \dfrac{6x^2y-3x^2-y^2}{2xy-2x^3}$; $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)} = \dfrac{5}{2}$

3. $y' = -\dfrac{b^2x}{a^2y}$

4. $y' = \dfrac{y\cos(xy)}{1-x\cos(xy)}$

5. $y' = \dfrac{2x-\sin(x+y)}{\cos y+\sin(x+y)}$

6. $y' = \dfrac{x^2\cos x+\sec y}{x\sec y\tan y}$

7. (a) Noting that $x^{\frac{2}{3}} = (\sqrt[3]{x})^2$ it follows that the values $x=-1$ and $y = 3\sqrt{3} = (\sqrt{3})^3$ simultaneously satisfy the equation.

   (b) Point-slope form: $y = \sqrt{3}(x+1)+3\sqrt{3}$, Slope-intercept form: $y = \sqrt{3}\,x+4\sqrt{3}$

3.12 Exercises (page 18)

1. $f'(x) = 20x^4+6x+1$

2. $\dfrac{dy}{dx} = 8x^7-\dfrac{6}{x^4}-\dfrac{1}{2}x^{-\frac{1}{2}}-2x^{-\frac{3}{2}}$

3. $g'(t) = \dfrac{2}{3}t^{-\frac{1}{3}}+4t^3-\dfrac{12}{t^3}$

4. $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}-\dfrac{5}{2}x^{-\frac{3}{2}}$

5. $g'(x) = \left(\dfrac{1}{2}x^{-\frac{1}{2}}+3\right)(x+\pi)+\left(\sqrt{x}+3x+1\right)$

6. $h'(y) = \dfrac{3(y+4)^2(1+0)(y+5)-(y+4)^3(1+0)}{(y+5)^2} = \dfrac{(y+4)^2(2y+11)}{(y+5)^2}$

7. $y' = \dfrac{1}{4}\left(x^3+2x+5\right)^{-\frac{3}{4}}\left(3x^2+2\right)$

8. $f'(\theta) = -3\sin(3\theta) + 2\sin\theta\cos\theta$

9. $g'(x) = 2x\sec^2\left(x^2+1\right)\cos(x) - \tan\left(x^2+1\right)\sin(x)$

10. $\dfrac{dy}{dt} = \dfrac{3}{4}(\sin t + 5)^{-\frac{1}{4}}\cos t$

11. $\dfrac{df}{dx} = \dfrac{1}{3}\left(\dfrac{x^4+5x-1}{x^2-3}\right)^{-\frac{2}{3}}\dfrac{\left(4x^3+5\right)\left(x^2-3\right)-\left(x^4+5x-1\right)(2x)}{\left(x^2-3\right)^2}$
$= \dfrac{1}{3}\left(\dfrac{x^4+5x-1}{x^2-3}\right)^{-\frac{2}{3}}\dfrac{2x^5-12x^3-5x^2+2x-15}{\left(x^2-3\right)^2}$

12. $y' = \dfrac{y-3x^2y^4}{4x^3y^3+2y-x}$

3.13 Exercises (page 18)

1. $\dfrac{d^2f}{dx^2} = 2\csc^2 x\cot x$

2. $f''(x) = 90(x-2)^8$; $f''(3) = 90$

3. $y'' = 6x\sec x + 6x^2\sec x\tan x + x^3\sec x\tan^2 x + x^3\sec^3 x$

4. $y'' = \dfrac{y-xy'}{y^2} = \dfrac{1}{y} - \dfrac{x^2}{y^3}$

3.14 Exercises (page 19)

1. $\dfrac{5}{\sqrt{18\pi}}$ km/h

2. $\dfrac{6}{25\pi}$ cm/sec

3. $\dfrac{3}{2}$ km/sec

4. $\dfrac{5}{3\pi}$ m/min

5. $\dfrac{1840}{29}$ km/h

6. $\dfrac{4}{5}$ m/sec

7. $-\dfrac{5}{4}$ cm/sec

3.15 Exercises (page 19)

1. $V \pm \Delta V = 125 \pm 15 \text{ cm}^3$
2. $V \pm \Delta V = 288\pi \pm 72\pi \text{ cm}^3$
3. $L(x) = 3 + 4(x - 2)$
4. $L(x) = 1 + 2(x - \pi/4)$

Module 3 Review Exercises (page 20)

1. $f'(x) = 3x^2$
2. $f'(x) = \dfrac{11}{(x+2)^2}$
3. $f'(x) = \dfrac{1}{\sqrt{2x+1}}$
4. $y' = 12x^3 + \dfrac{\sqrt[3]{2}}{3}x^{-2/3} + \dfrac{5}{2}x^{-3/2}$
5. $g'(x) = \left(\dfrac{1}{\sqrt{2x}} - 4\right)(3x + \sin x) + \left(\sqrt{2x} - 4x + 3\right)(3 + \cos x)$
6. $h'(y) = -\dfrac{3y + 28}{2\sqrt{y+5}\,(3y+2)^2}$
7. $f'(\theta) = -2\cos\theta\sin\theta - 8\theta\sin\left(\theta^2\right)$
8. $g'(x) = 3x^2\sec\left(x^3+4\right)\tan\left(x^3+4\right)\cos(2x) - 2\sec\left(x^3+4\right)\sin(2x)$
9. $f'(x) = \dfrac{1}{5}\left(\dfrac{x^3 - 4x + 10}{4x^2+5}\right)^{-4/5}\dfrac{4x^4 + 31x^2 - 80x - 20}{\left(4x^2+5\right)^2}$
10. $\dfrac{dy}{dx} = \dfrac{y - 4x^3y^3}{3x^4y^2 + 8y - x - \cos y}$
11. Point-slope form: $y = -6(x - \pi/2) + 3$, Slope-intercept form: $y = -6x + 3\pi + 3$
12. $x = 1$, $x = -3$
13. $\theta = \dfrac{\pi}{3} + 2n\pi$, $\theta = \dfrac{5\pi}{3} + 2n\pi$, $\theta = n\pi$ (n an integer)
14. $57\pi \text{ cm}^3/\text{sec}$
15. $-\dfrac{1}{10}$ rad/sec
16. $\dfrac{1}{2(3)^{1/4}}$ cm/sec

4.1 Exercises (page 21)

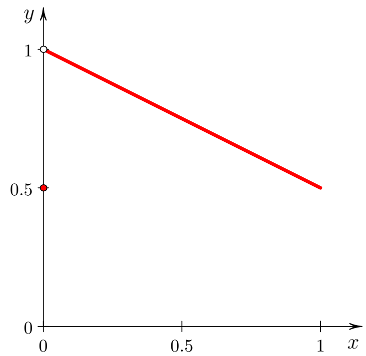
1. Note: It is assumed that the functions extend beyond the graph of the plot with the same trend unless they are explicitly terminated with a point. The values below are approximate.
   (a) Relative minima: $f(-1.4) = -0.8$ and $f(1.9) = 0.5$; Relative maxima: $f(-2.0) = 2.0$, $f(0.4) = 1.0$, and $f(1.3) = 2.6$. No absolute minimum nor absolute maximum.
   (b) Absolute maximum value of $f(0) = 0.5$. This is also a relative maximum.
   (c) Relative minimum: $f(-1) = 2$; Relative maximum: $f(-3) = 6$, $f(0) = 3$. Absolute maximum: $f(-3) = 6$, No absolute minimum.
2. $x = 2, 4$
3. $x = 0$
4. $s = \sqrt{6}$
5. $x = -\frac{1}{3},\ -3$
6. $t = 1$
7. No critical numbers
8. $t = -2,\ 2$
9. $x = -\sqrt{5},\ 0,\ \sqrt{5}$
10. θ in $\left\{\frac{\pi}{3} + 2n\pi \mid n \text{ an integer}\right\} \bigcup \left\{\frac{5\pi}{3} + 2n\pi \mid n \text{ an integer}\right\}$
11. Absolute maximum: $f(-2) = f(2) = 13$, Absolute minimum: $f\left(\pm\frac{1}{\sqrt{2}}\right) = \frac{3}{4}$
12. Absolute maximum: $f(-1) = 2$, Absolute minimum: $f\left(-1/3\right) = \frac{14}{27}$
13. Absolute maximum: $g(0) = 0$, Absolute minimum: $g\left(2/3\right) = -\frac{4}{3}\sqrt{\frac{2}{3}}$
14. Absolute maximum: $H(8) = -1$, Absolute minimum: $H\left(-1\right) = -4$
15. Absolute maximum: $f(\pi/4) = f(5\pi/4) = \frac{1}{2}$, Absolute minimum: $f(3\pi/4) = f(7\pi/4) = -\frac{1}{2}$

4.2 Exercises (page 22)

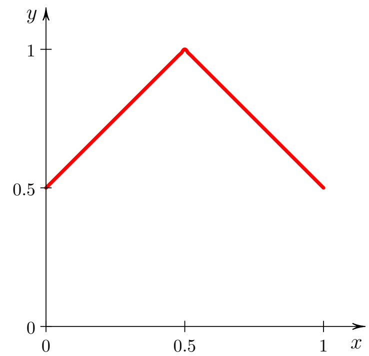
1. f is continuous on $[-1,4]$, differentiable on $(-1,4)$, and $f(-1)=f(4)=2$ so Rolle's Theorem applies. Solving $f'(c)=0$ shows $c=7/3$ or $c=-1$, however only $c=7/3$ is in the open interval $(-1,4)$.

2. Each of the following graphs of f give one possible counterexample. There is no point in the interval $[0,1]$ where the function has a horizontal tangent and hence the conclusion to Rolle's theorem is invalid.

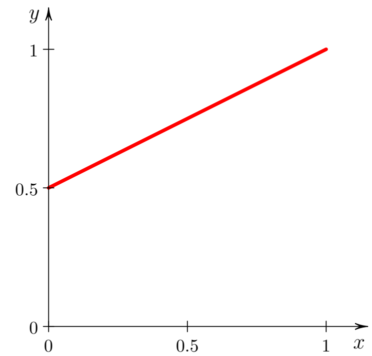
   (a) Continuity fails:



   (b) Differentiability fails:



   (c) $f(0)\neq f(1)$:



3. Use the Mean Value Theorem.

4. $c=1$ in $(-1,2)$ has $f'(c)=\dfrac{10-(-5)}{2-(-1)}=5$.

4.3 Exercises (page 23)

1. Decreasing on: $(-\infty,-1)$; Increasing on: $(-1,\infty)$; No relative maxima; Relative minimum: $f(-1)=0$

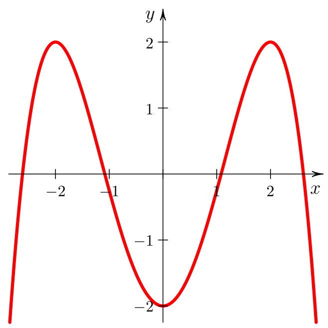
2. Increasing on: $(-\infty,1)\cup(1,\infty)$; No relative maxima or minima

3. Decreasing on: $\left(0,\dfrac{7\pi}{6}\right)\cup\left(\dfrac{11\pi}{6},2\pi\right)$; Increasing on: $\left(\dfrac{7\pi}{6},\dfrac{11\pi}{6}\right)$;
   Relative maximum: $f\left(\dfrac{11\pi}{6}\right)=2\sqrt{3}-\dfrac{11\pi}{3}$; Relative minimum: $f\left(\dfrac{7\pi}{6}\right)=-2\sqrt{3}-\dfrac{7\pi}{3}$

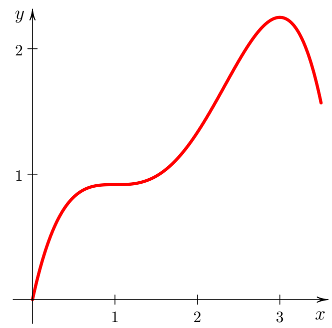
4. Decreasing on: $(-\infty,0)$; Increasing on: $(0,\infty)$; Relative minimum: $f(0)=0$

5. Decreasing on: $(-\infty,0)$; Increasing on: $(0,\infty)$; Relative maximum: $f(0)=2$. Note that the First Derivative Test cannot be applied here because $f(x)$ is discontinuous at $x=0$. One must return to the definition of relative maximum to evaluate the critical number $x=0$.

6.



7.



Note that your graphs should be equivalent up to vertical shift by a constant.

4.4 Exercises (page 23)

1. Notice that $f''(x) = 24x^2$ and $24x^2$ is positive for all values of x except 0. The only potential inflection point would therefore occur at $x = 0$ but the concavity is positive on both sides of $x = 0$ and hence does not change at that value.

2. Notice that $f''(x) = \dfrac{2}{x^3}$ which is negative for $x < 0$ and positive for $x > 0$. Therefore the concavity does, in fact, change at $x = 0$. However the function is not defined at 0 so there is no point on the curve there (it is a vertical asymptote) and hence no inflection point exists.

3. Notice that $f'(x) = 5x^2(x-3)(x+3)$, $f''(x) = 10x\left(\sqrt{2}x-3\right)\left(\sqrt{2}x+3\right)$. Relative maximum: $f(-3) = 163$; Relative minimum: $f(3) = -161$; Inflection points: $\left(-\dfrac{3}{\sqrt{2}}, -\dfrac{243}{2^{\frac{5}{2}}} + \dfrac{405}{2^{\frac{3}{2}}} + 1\right)$, $(0,1)$, $\left(\dfrac{3}{\sqrt{2}}, \dfrac{243}{2^{\frac{5}{2}}} - \dfrac{405}{2^{\frac{3}{2}}} + 1\right)$; Concave upward on: $\left(-3/\sqrt{2}, 0\right) \cup \left(3/\sqrt{2}, \infty\right)$; Concave downward on: $\left(-\infty, -3/\sqrt{2}\right) \cup \left(0, 3/\sqrt{2}\right)$;

4.5 Exercises (page 24)

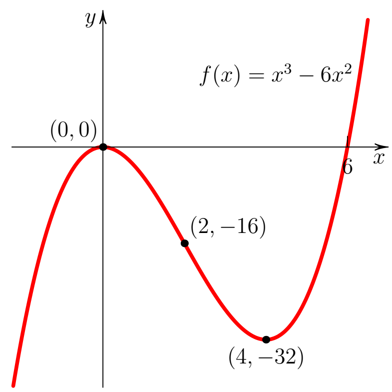
1. Relative maximum: $f(-2) = 17$; Relative mininum: $f(2) = -15$

2. Relative maximum at $f\left(-\dfrac{\pi}{2}\right) = 3$; Relative minimum: $f\left(\dfrac{\pi}{2}\right) = -5$;

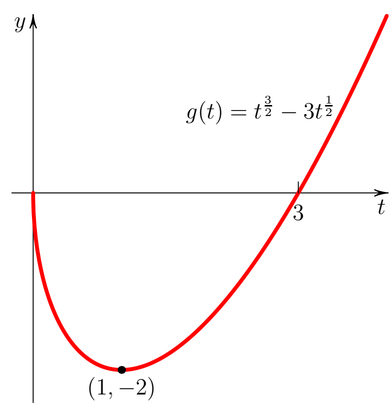
3. No since $f''(x) = 12\cos^2 x \sin^2 x - 4\sin^4 x$ vanishes at $x = 0$ so the test is inconclusive. The first derivative test shows that $x = 0$ is a the location of a local minimum of f.

4.6 Exercises (page 24)

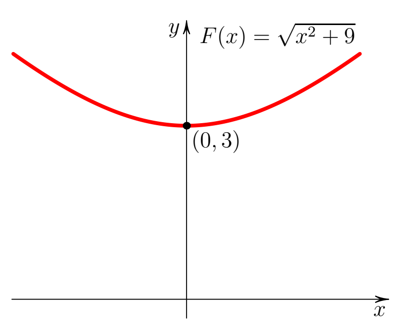
1. $D = \mathbb{R} = (-\infty, \infty)$; x-int= 0, 6; y-int= 0; Increasing on: $(-\infty, 0) \cup (4, \infty)$; Decreasing on: $(0, 4)$; Relative maximum: $f(0) = 0$; Relative minimum: $f(4) = -32$; Concave upward on: $(2, \infty)$; Concave downward on: $(-\infty, 2)$; Inflection point: $(2, -16)$; Graph:



2. $D = [0, \infty)$; t-int= 0, 3; y-int= 0; Increasing on: $(1, \infty)$; Decreasing on: $(0, 1)$; No relative maxima; Relative minimum: $g(1) = -2$; Concave upward on: $(0, \infty)$; Not concave downward anywhere; No inflection points; Graph:



3. $D = \mathbb{R} = (-\infty, \infty)$; No x-int; y-int= 3; Increasing on: $(0, \infty)$; Decreasing on: $(-\infty, 0)$; No relative maxima; Relative minimum: $F(0) = 3$; Concave upward on: $(-\infty, \infty)$; Not concave downward anywhere; No inflection points; Graph:



4.7 Exercises (page 24)

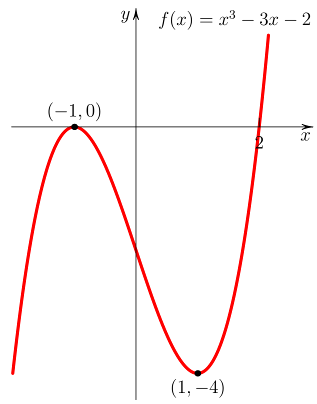
1. 0
2. ∞
3. $-\dfrac{1}{6}$
4. 2
5. $-\dfrac{3}{2}$
6. $\dfrac{1}{3}$
7. ∞
8. 0
9. 1
10. $-\sqrt{2}$
11. 4
12. 2
13. 3
14. Horizontal asymptote: $y = \dfrac{3}{2}$
15. No horizontal asymptotes
16. Horizontal asymptotes: $y = -1$, $y = 1$
17. Horizontal asymptote: $y = \dfrac{1}{2}$
18. Horizontal asymptote: $y = 0$ (Hint: The Squeeze Theorem, generalized to a limit at infinity, can be used here to evaluate the limits.)
19. Horizontal asymptote: $y = 5$
20. Horizontal asymptote: $y = 1$
21. Horizontal asymptotes: $y = -\dfrac{1}{2}$, $y = \dfrac{1}{2}$

4.8 Exercises (page 25)

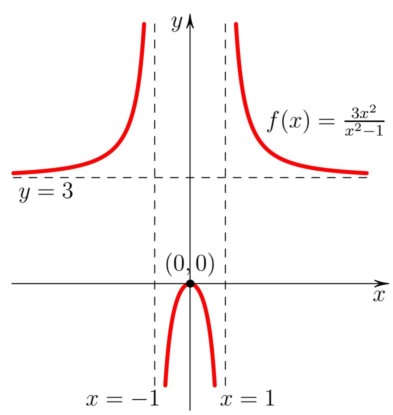
1. Slant asymptote: $y = 3x - 10$
2. No slant asymptotes
3. Slant asymptote: $y = x - 2$

4.9 Exercises (page 25)

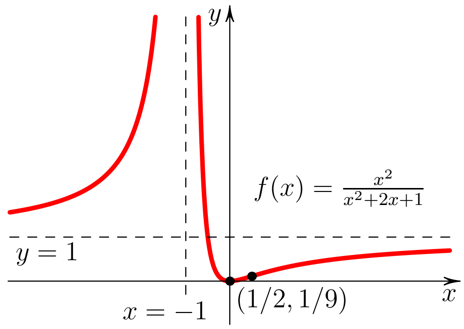
1. Notice $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$ and $f''(x) = 6x$. $D = \mathbb{R} = (-\infty, \infty)$; x-int$= -1,\ 2$; y-int$= -2$; No asymptotes; No symmetry; Increasing on: $(-\infty, -1) \cup (1, \infty)$; Decreasing on: $(-1, 1)$; Relative maxima: $f(-1) = 0$; Relative minimum: $f(1) = -4$; Concave upward on: $(0, \infty)$; Concave downward on $(-\infty, 0)$; Inflection point: $(0, -2)$; Graph:



2. Notice that the first and second derivatives simplify to $y'(x) = -\dfrac{6x}{(x^2-1)^2}$ and $y'' = \dfrac{6(3x^2+1)}{(x^2-1)^3}$. $D = \mathbb{R} - \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$; x-int$= 0$; y-int$= 0$; Horizontal asymptote: $y = 3$; Vertical asymptotes: $x = -1$, $x = 1$; Symmetric about the y-axis; Increasing on: $(-\infty, -1) \cup (-1, 0)$; Decreasing on: $(0, 1) \cup (1, \infty)$; Relative maxima: $f(0) = 0$; No relative minimum; Concave upward on: $(-\infty, -1) \cup (1, \infty)$; Concave downward on $(-1, 1)$; No inflection points; Graph:



3. Notice that the derivatives simplify to $f'(x) = \dfrac{2x}{(x+1)^3}$ and $f''(x) = \dfrac{2-4x}{(x+1)^4}$. $D = \mathbb{R} - \{-1\} = (-\infty, -1) \cup (-1, \infty)$; x-int$= 0$; y-int$= 0$; Horizontal asymptote: $y = 1$; Vertical asymptote: $x = -1$; No symmetry; Increasing on: $(-\infty, -1) \cup (0, \infty)$; Decreasing on: $(-1, 0)$; No relative maxima; Relative minimum: $f(0) = 0$; Concave upward on: $(-\infty, -1) \cup (-1, 1/2)$; Concave downward on $(1/2, \infty)$; Inflection point: $(1/2, 1/9)$; Graph:



4.10 Exercises (page 26)

1. Base length=$\frac{5}{2}$ m, Height=$\frac{5}{2}$ m, Area=$\frac{25}{8}$ m^2

2. $(3/5, 16/5)$. For part (a) note it is easier to minimize the distance-squared than the distance. For part (b) the line perpendicular is $y = -\frac{1}{2}(x-3)+2 = -\frac{1}{2}x+\frac{1}{2}$. To find the intersection of this and the original line we solve the two equations simultaneously since the point of interest must lie on both lines.

3. First number=10, Second number=5

4. Radius= $\sqrt[3]{\frac{5}{\pi}} \approx 1.17$ cm, Height= $2\sqrt[3]{\frac{5}{\pi}} \approx 2.34$ cm

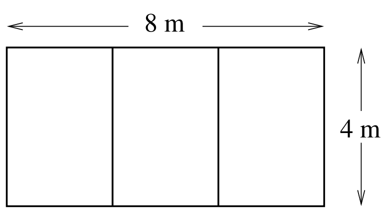
5. (a) $x = \dfrac{2}{3\sqrt{7}} \approx 0.252$ km

   (b) $x = \dfrac{2}{\sqrt{15}} \approx 0.516$ km

   (c) $x = \dfrac{b}{\sqrt{\left(\frac{w}{v}\right)^2 - 1}}$

   (d) Although the total time taken t does depend on a the value for x does not. As expected, it does not matter how far upstream you start, you would still turn off at the same location. That said, a does play a role in the solution because valid values of x must lie in the interval $[0, a]$. If a had been smaller than the solution for x, say in part (a) had the starting distance been 0.2 km, then the critical number would no longer be in the interval and that solution would be invalid. One would consider the endpoints of the interval to see which was optimal.

6. $x = 40$ m, $h = 20$ m

7. 

8. $10\left(2-\sqrt{2}\right) \approx 5.9$ km

9. $x = 100$ m, $y = \dfrac{200}{\pi}$ m

10. $r = 4$ cm, $\theta = 2$ rad $\approx 115°$, $A = 16$ cm^2

11. (a) Radius $r = \sqrt{\dfrac{A}{\pi\sqrt{3}}}$, Height $h = \sqrt{\dfrac{2A}{\pi\sqrt{3}}}$, $\dfrac{h}{r} = \sqrt{2}$

    (b) Let $\alpha = n\tan\dfrac{\pi}{n}$. Apothem $H = \sqrt{\dfrac{A}{\alpha\sqrt{3}}}$, Height $h = \sqrt{\dfrac{2A}{\alpha\sqrt{3}}}$, $\dfrac{h}{H} = \sqrt{2}$
    As $n \to \infty$ the apothem H approaches the radius r of the circle. The limit

$$\lim_{n\to\infty} \alpha = \lim_{n\to\infty} n\tan\frac{\pi}{n} = \lim_{n\to\infty} \pi \frac{1}{\cos(\pi/n)} \frac{\sin(\pi/n)}{(\pi/n)} = \pi \cdot \frac{1}{1} \cdot 1 = \pi\,,$$

    since $\pi/n \to 0$ as $n \to \infty$, thus showing part (b) gives part (a) in the large n limit. Note that from observation a typical tipi has a height/radius ratio of about $\sqrt{2} \approx 1.4$.

Module 4 Review Exercises (page 29)

1. $x = -1$, $x = -3$

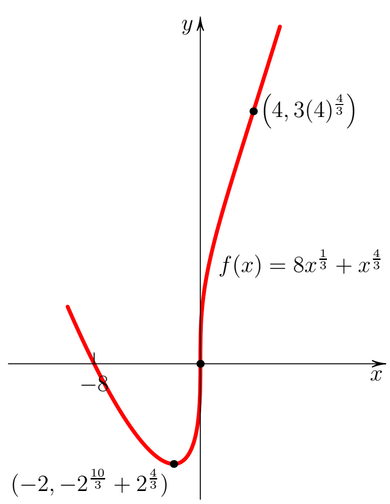
2. $t = 0$, $t = 3$

3. θ in $\left\{\dfrac{\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{5\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{\pi}{2} + 2n\pi\right\} \cup \left\{\dfrac{3\pi}{2} + 2n\pi\right\}$ (n an integer)

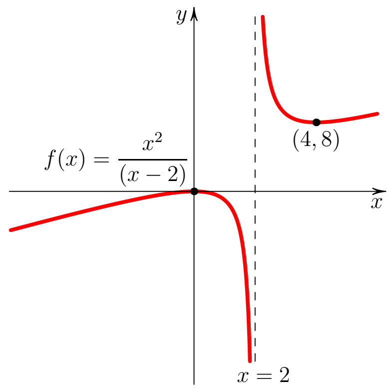
4. Absolute maximum: $f(1) = \dfrac{1}{17}$, Absolute mininimum: $f(-1) = -\dfrac{1}{17}$

5. Absolute maximum: $g(1) = \sqrt{7}$, Absolute minimum: $g(0) = 0$

6. $D = \mathbb{R} = (-\infty, \infty)$; x-int=0, -8; y-int=0; No asymptotes; Increasing on: $(-2, \infty)$; Decreasing on: $(-\infty, -2)$; No relative maxima; Relative mininimum: $f(-2) = -2^{\frac{10}{3}} + 2^{\frac{4}{3}}$; Concave upward on $(-\infty, 0) \cup (4, \infty)$; Concave downward on $(0, 4)$; Inflection points: $(0, 0)$, $\left(4, 3(4)^{\frac{4}{3}}\right)$; Graph:



7. $D = \mathbb{R} - \{2\} = (-\infty, 2) \cup (2, \infty)$; x-int=0; y-int=0; Vertical asymptote: $x = 2$; Increasing on: $(-\infty, 0) \cup (4, \infty)$; Decreasing on $(0, 2) \cup (2, \infty)$; Relative maximum: $f(0) = 0$; Relative minimum: $f(4) = 8$, , Concave upward on $(2, \infty)$; Concave downward on $(-\infty, 2)$; No inflection points; Graph:



8. 0

9. 5

10. $-\dfrac{\sqrt{5}}{2}$

11. $-\infty$

12. Vertical asymptote: $x = 2$, Horizontal asymptote: $y = 2$

13. Vertical asymptote: $t = -\dfrac{3}{2}$, Horizontal asymptotes: $y = -1$, $y = 1$

14. Vertical length in diagram is 30 m and horizontal length is $\dfrac{100}{3}$ m; lot area=2560 m^2

15. Brick side is 10 m and exclusively fence side is 25 m

16. Maximal area occurs when Width=Circle diameter=$\dfrac{20}{4+\pi}$ m and Height=$\dfrac{10}{4+\pi}$ m (i.e. half the width).

5.1 Exercises (page 31)

1. Both F_1 and F_2 differentiate to x^3. As this problem suggests, any two antiderivatives of a function differ at most by a constant.

2. $F(x) = x^3 - \dfrac{5}{2}x^2 + 6x + C$

3. $F(x) = \dfrac{1}{2}x^2 - \dfrac{4}{x} + C$

4. $G(t) = \dfrac{2}{3}t^{\frac{3}{2}} + 4t^{\frac{1}{2}} + C$

5. $H(x) = \dfrac{3}{5}x^{\frac{5}{3}} - \dfrac{4}{7}x^7 + \pi x + C$

6. $F(\theta) = 2\sin\theta + \cos\theta + \tan\theta + C$

7. $f(x) = \dfrac{1}{10}x^5 - \dfrac{5}{3}x^3 + \dfrac{3}{2}x^2 + Cx + D$

8. $f(t) = \frac{4}{15}t^{\frac{5}{2}} + t^3 - \frac{5}{3}t + \frac{7}{5}$
9. $f(\theta) = -3\sin\theta - \cos\theta + \frac{5}{2}\theta^2 + 2\theta + 4$
10. If $f'''(x) = 0$, then $f''(x) = C$ where C is a real constant. If C is non-zero then f has the same concavity everywhere, while if $C = 0$ then $f''(x) = 0$ implies $f(x) = Dx + E$ so f is linear and hence has no point of inflection.

5.2 Exercises (page 32)

1. $\frac{41}{4}$
2. 60
3. $\frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6} + n\right) = \frac{2n^2+3n+7}{6n^2}$
4. $\frac{2n^3+3n^2+n}{6} - \frac{3n^2+3n}{2} = \frac{n^3-3n^2-4n}{3}$

5.3 Exercises (page 32)

1. (a) $S_n = \frac{3}{n}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right) = \sum_{i=1}^{n}\left(\frac{9}{n} + \frac{27i}{n^2} + \frac{54i^2}{n^3}\right) = \frac{81}{2} + \frac{81}{2n} + \frac{9}{n^2}$

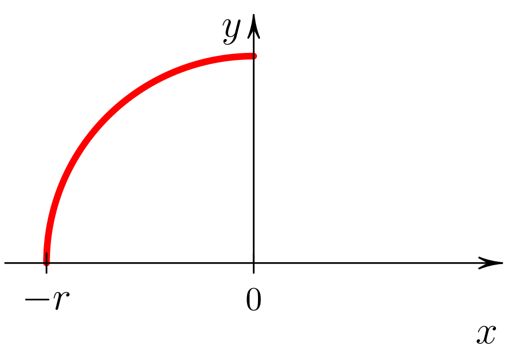
   (b) $A = \lim_{n\to\infty} S_n = \frac{81}{2}$

5.4 Exercises (page 33)

1. (a) i. -6
      ii. 4

   (b) i. Increasing on $(-6,-5)\cup(0,5)$
      ii. Decreasing on $(-5,0)\cup(5,6)$

2. 6
3. 30
4. -3
5. $\frac{7}{2}$
6. Notice that the area can be viewed as the quarter of a circle:



Then the integral is $\frac{1}{4}\left(\pi r^2\right)$. (Since the area is above the x-axis the integral is positive.)

7. $\int_3^9 f(x)\,dx$

8. Since $-1 \le \sin x \le 1$ it follows (adding 2 to these inequalities) that $m = 1 \le 2 + \sin x \le 3 = M$. Since $b = 1$ and $a = -1$ we have that $m(b-a) = 1(2) = 2$ and $M(b-a) = 3(2) = 6$.

9. $S_n = \dfrac{3}{n}\left(\dfrac{27n^4 + 54n^3 + 27n^2}{4n^3} + n\right) = \dfrac{93n^2 + 162n + 81}{4n^2}$
and so $\displaystyle\int_0^3 \left(x^3 + 1\right) dx = \lim_{n\to\infty} S_n = \frac{93}{4}$.

10. $S_n = b^3\left(\dfrac{1}{3} + \dfrac{1}{2n} + \dfrac{1}{6n^2}\right)$ and so $\displaystyle\int_0^b x^2\,dx = \lim_{n\to\infty} S_n = \frac{b^3}{3}$.

5.5 Exercises (page 34)

1. $\dfrac{dF}{dx} = \sqrt{x^3 + 2x + 1}$

2. $h'(x) = 4x^3\sqrt{x^{12} + 2x^4 + 1}$

3. $g'(x) = -\left[\cos\left(x^3\right)\right]$

4. Noting that $H(x) = \int_{2x}^0 \sqrt[3]{t^3+1}\,dt + \int_0^{3x} \sqrt[3]{t^3+1}\,dt = -\int_0^{2x} \sqrt[3]{t^3+1}\,dt + \int_0^{3x} \sqrt[3]{t^3+1}\,dt$, one gets $H'(x) = -2\sqrt[3]{8x^3+1} + 3\sqrt[3]{27x^3+1}$

5. $f'(x) = \dfrac{2}{\sqrt{\pi}}e^{-(x^3)^2}\left(3x^2\right) = \dfrac{6x^2}{\sqrt{\pi}}e^{-x^6}$ (Use the Chain Rule and the Fundamental Theorem of Calculus.)

6. $(x, y) = \left(\frac{3}{2}\text{ km}, \frac{7}{16}\text{ km}\right)$

5.6 Exercises (page 35)

1. $\dfrac{44}{3}$

2. $-\dfrac{14}{3}$

3. 2

4. $\dfrac{7}{6}$

5. 0

6. $\dfrac{13}{2}$

7. The Fundamental Theorem of Calculus is not applicable here because the integrand $\frac{1}{x^2}$ is discontinuous on $[-1, 1]$. The area under the curve (i.e. the integral) in fact diverges to $+\infty$. To show that requires a consideration of improper integrals.

5.7 Exercises (page 35)

1. The Fundamental Theorem of Calculus shows that the definite integral $\int_a^b f(x)\,dx$ is, assuming the conditions of the theorem are met, intimately connected to the antiderivative of f by the relation $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is an antiderivative of f. Thus in many cases finding a definite integral is a two-step process where first one finds an antiderivative of f and then secondly takes the difference of that function evaluated at the limits of integration. It is natural, therefore, to generally write the answer to the first step, namely the antiderivative of f, symbolically as $\int f(x)\,dx$. (The notation is further convenient because it embeds the function we are antidifferentiating directly in the symbol in the same way we write $\frac{df}{dx}$ abstractly for the derivative of f.)

2. $\dfrac{x^4}{4} - \dfrac{3x^5}{5} - 6x + C$

3. $4x^{\frac{1}{2}} + \dfrac{2}{3}x^{\frac{3}{2}} + C$

4. $-\csc\theta + C$

5. $\displaystyle\int \left(\tan^2 x + 1\right)\,dx = \int \sec^2 x\,dx = \tan x + C$

6. The equation states that the derivative of y is $x^2 + 9$ so y must be the general form of the antiderivative of $x^2 + 9$ which is just its indefinite integral:

   $\displaystyle y = \int \left(x^2 + 9\right)\,dx = \frac{1}{3}x^3 + 9x + C$

   The general solution for any differential equation of the form $y' = f(x)$ is similarly $y = \int f(x)\,dx$.

5.8 Exercises (page 35)

1. $-\dfrac{1}{12}\left(x^3 + 3x^2 + 4\right)^{-4} + C$

2. $\dfrac{1}{3}\left(5x^2 + 2x\right)^{\frac{3}{2}} + C$

3. $2\sin(\sqrt{t}) + \frac{1}{4}t^4 + C$

4. $-\dfrac{2}{3}(3 - \sin\theta)^{\frac{3}{2}} + C$

5. $\dfrac{1}{40}(4x + 1)^{\frac{5}{2}} - \dfrac{1}{24}(4x + 1)^{\frac{3}{2}} + C$

6. $\dfrac{1}{2}\tan\left(2x - \dfrac{\pi}{3}\right) + C$

7. Using $u = x^4 + 9$, integral is $\displaystyle = \int_9^{25} u^{\frac{1}{2}}\,\frac{du}{4} = \frac{49}{3}$

8. Using $u = \tan\theta$, integral is $\displaystyle = \int_0^1 u^4\,du = \frac{1}{5}$

9. Breaking the integral into three separate integrals (one per term) and using $u = 1 - x$ on the last one that integral equals $= \int_1^0 u^5(-du) = \frac{1}{6}$. Combining this with the definite integral of the first two terms gives the final answer $\frac{5}{3}$. Alternatively, one can find the indefinite integral of the third term (i.e. substitute back to x) to get the antiderivative of the entire integrand as $x + \frac{1}{2}x^2 - \frac{1}{6}(1-x)^6 + C$ and evaluate that at the original limits of x.

10. Using $u = 2x^2 + 1$, integral is $= \displaystyle\int_3^{19} u^{-2}\frac{du}{4} = \frac{4}{57}$

11. Using $u = \cos t$, integral is $= \displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} u^{-\frac{2}{3}}(-du) = 3\left(\frac{1}{\sqrt[6]{2}} - \frac{1}{\sqrt[3]{2}}\right)$

12. Using $u = \dfrac{2\pi t}{T}$, integral is $= \displaystyle\int_0^{\frac{\pi}{3}} \cos(u)\frac{T\,du}{2\pi} = \frac{\sqrt{3}\,T}{4\pi}$

5.9 Exercises (page 36)

1. $\dfrac{1}{4}x^4 + \dfrac{2}{3}x^{\frac{3}{2}} + \dfrac{1}{x} + 5x + C$

2. $\dfrac{1}{7}x^7 + x^4 + 4x + C$

3. $\dfrac{8}{5}x^{\frac{5}{2}} + 2x^2 + \dfrac{2}{3}x^{\frac{3}{2}} + C$

4. $\sin\theta + \tan\theta + C$

5. $\dfrac{1}{6}\left(x^3 + 4\right)^6 + C$

6. $\dfrac{1}{11}\left(\sin\theta + 3\right)^{11} + C$

7. $\dfrac{6}{7}\left(\sqrt{t} + 7\right)^{\frac{7}{3}} + C$

8. $\dfrac{16806}{10}$

9. $\dfrac{5}{216}$

10. $\dfrac{1}{2}$

11. 0 Note that since sine and tangent are odd, the integrand itself is odd. Since the limits are $\pm a$, the integral vanishes.

5.10 Exercises (page 36)

1. $\dfrac{39}{2}$ units^2
2. 13 units^2
3. $\dfrac{32}{3}$ unit^2
4. $\dfrac{32}{3}$ unit^2
5. $\dfrac{1}{6}$ unit^2
6. 3 unit^2
7. $\dfrac{16}{3}$ units^2

5.11 Exercises (page 37)

1. (a) $\displaystyle\int_0^1 v(t)\,dt = \frac{2}{\pi}$ cm

   (b) $\displaystyle\int_1^2 v(t)\,dt = -\frac{2}{\pi}$ cm

   (c) 0 cm (The particle is back where it started after 2 seconds.)

2. $\dfrac{8750}{9}$ gigalitres

Module 5 Review Exercises (page 38)

1. $F(x) = \dfrac{5}{8}x^{8/5} + 8x^{1/2} + \dfrac{1}{4}x^4 + 10x + C$
2. $G(x) = -x^{-1/2} - \dfrac{5}{2x} - \dfrac{1}{4x^2} + C$
3. $F(\theta) = -3\cos\theta + 5\sin\theta + \dfrac{1}{4}\theta^4 + \theta + C$
4. $G(\theta) = 2\sec\theta - 2\sin\theta + \tan\theta + C$
5. $f(x) = \dfrac{4}{15}x^{5/2} + \dfrac{1}{12}x^4 - 3x^2 + Cx + D$
6. $f(t) = \dfrac{9}{2}t^{8/3} - \dfrac{1}{2}t^3 - \dfrac{5}{2}t^2 - \dfrac{13}{2}t + 6$
7. $f(\theta) = -5\sin\theta + 4\cos\theta + 5\theta^2 + 7\theta - 17$
8. $\dfrac{1}{45}\left(x^5+3\right)^9 + C$
9. $\dfrac{1}{12}\left[\tan(2\theta)+1\right]^6 + C$
10. $\dfrac{15}{8}\left(t^{2/5}-4\right)^{4/3} + C$
11. $-\dfrac{96}{7}$

12. $\frac{1}{15}\left(\frac{\sqrt{2}}{8}+1\right)$

13. $\frac{\pi}{6}$

14. $\frac{32}{3}$

15. $\frac{27}{20}$

16. $\frac{64}{3}$

17. $y = -\frac{3}{x} + \frac{1}{4}x^4 + C$

18. $y = 2\sin x + 3$

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