DEPARTMENT OF MATHEMATICS AND STATISTICS

Math122–001, 002 Linear Algebra I

Final Examination, Semester 2012–10

14 April 2012

Instructors: R. F. Bailey (001), C.-H. Guo (002)

Answer all the questions

Time: 3 hours

Family name: ____________________ **First name:** ____________________

Student ID: ____________________ **Section:** 001 (Bailey) 002 (Guo)

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Marks available:	10	9	12	10	8	6	10	12	6	11	6	100
Marks:												

READ THESE INSTRUCTIONS CAREFULLY

1. This examination has 11 questions.
2. You have 3 hours to complete this examination.
3. This is a closed book examination, and no notes of any kind are allowed.
4. The use of calculators is allowed; however the calculator must meet the guidelines described in the calculator policy of the Department of Mathematics and Statistics.
5. Cell phones, pagers, or any text storage or communication devices are not permitted. **Please turn off your cell phones!**
6. Read each question carefully.
7. Where it is possible to check your work, do so.
8. Good luck!

DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE BEEN TOLD TO DO SO

1. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} x & - & 2y & - & 3z & & & = & -1 \\ & & y & + & 2z & + & 2w & = & 6 \\ 2x & - & 5y & - & 8z & - & w & = & -6 \end{array}$$

(a) (6 marks) Find the general solution of this system.

(b) (2 marks) Find a solution (x, y, z, w) with $z = 5$, if any.

(c) (2 marks) Does there exist a solution with $w = 1$?

2. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & + & 2x_4 & = & 1 \\ x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 3 \\ & & 2x_2 & + & 2x_3 & + & px_4 & = & q \end{array}$$

where p, q are some arbitrary real numbers.

(a) (1 mark) Write down the augmented matrix for this system.

(b) (8 marks) Find all values of p, q so that the system has

i. a unique solution;

ii. no solution;

iii. infinitely many solutions.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & a \\ a & 0 & 0 & 5 \end{bmatrix},$$

where a is a real number.

(a) (4 marks) Show that the matrix A is invertible if and only if $a^2 - 5 \neq 0$.

(b) (5 marks) Find the inverse of A when $a = 2$.

(c) (3 marks) Use your answer to part (b) to solve the linear system

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$

4. (a) (4 marks) Find the parametric equation of the line ℓ_1 in $\mathbb{R}^3$ which passes through the point $P = (1, 0, 1)$ and which is orthogonal to the vectors $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (1, -1, 4)$.

(b) (2 marks) Find the equation of the plane in $\mathbb{R}^3$ which is orthogonal to ℓ_1 and which passes through the point $Q = (2, 2, 2)$.

(c) (4 marks) Find the intersection of the plane you found in part (b) with the line ℓ_2 whose parametric equation is

$$x = 1 + t, \quad y = 2 - 2t, \quad z = -t.$$

5. (a) (4 marks) Consider the set of vectors $V = \{(x,y,z) \in \mathbb{R}^3 \mid z = x+2y\}$ (that is, V is the set of all vectors (x,y,z) in $\mathbb{R}^3$ such that $z = x+2y$.) Is V a subspace of $\mathbb{R}^3$? Explain your answer.

(b) (4 marks) Consider the set of vectors $W = \{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$. Is W a subspace of $\mathbb{R}^2$? Explain your answer.

6. (6 marks) For each of the following sets of vectors, determine whether or not it is a basis for $\mathbb{R}^3$. Explain your answers.

   (a) $(1,1,0)$, $(1,0,3)$, $(2,1,4)$;

   (b) $(1,1,0)$, $(1,3,-1)$, $(1,-3,2)$;

   (c) $(1,0,3)$, $(1,-1,5)$, $(-1,1,0)$, $(0,3,1)$, $(7,7,8)$.

7. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 \\ -1 & 2 & 5 & 3 \\ 2 & -4 & -1 & 3 \end{bmatrix}.$$

(a) (4 marks) Find the row-reduced echelon form of A.

(b) (3 marks) Find a basis for, and the dimension of, the null space of A.

(c) (3 marks) Find a basis for, and the dimension of, the column space of A.

8. You are told that a subspace U of $\mathbb{R}^4$ has a basis

$$\{(1,1,1,1),\,(0,2,-1,-1),\,(1,2,0,-2)\}.$$

(a) (8 marks) Starting with the basis given above, use the Gram–Schmidt Process to find an orthonormal basis for U.

(b) (4 marks) You are also told that the vector $\mathbf{w} = (0,1,0,2)$ is in the subspace U. Use you answer to part (a) to express $\mathbf{w}$ as a linear combination of the vectors in your orthonormal basis.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x,y,z) = (2x+y, x-y+2z, 3y+5z).$$

(a) (2 marks) Give the matrix representation of T.

(b) (4 marks) Is T one-to-one? Is T invertible? Explain your answers.

10. Consider the matrix

$$A = \begin{bmatrix} 4 & -3 & 3 \\ 2 & -3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) (5 marks) Find the characteristic polynomial of A and give all eigenvalues of A.

(b) (6 marks) For each eigenvalue of A, find a basis for the corresponding eigenspace.

11. Suppose A is an $n \times n$ matrix.

(a) (3 marks) Show that, if 0 is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$, then $\mathbf{x}$ is in the null space of A.

(b) (3 marks) Show further that, if 0 is an eigenvalue of A, then A is not invertible.

END OF QUESTIONS