The Majorization and Trumping Orders in Quantum Information

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Majorization

Introduced by R. Muirhead (mathematician, physicist, engineer) in 1903, developed by Lorenz (1905), an economist interested in inequalities of wealth and inequalities of income.

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Majorization: definition

Definition

Let $x = (x_1, x_2, ..., x_d), y = (y_1, y_2, ..., y_d) \in \mathbb{R}^d$. We say that x is majorized by y, written $x \prec y$, if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow} \quad 1 \leq k \leq d, \text{ where } x_i^{\downarrow} \geq x_{i+1}^{\downarrow},$$

with equality when $k = d$.

If equality does not necessarily hold for $k = d$, we say that x is *sub-majorized* by y and we write $x \prec_w y$, where the w stands for "weak".

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Equivalently: $x \prec y$ if

$$\sum_{j=1}^{k} x_j^{\uparrow} \geq \sum_{j=1}^{k} y_j^{\uparrow} \quad 1 \leq k \leq d, \text{ where } x_i^{\uparrow} \leq x_{i+1}^{\downarrow},$$

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Examples

For d-dimensional vectors,

$$\begin{aligned}\left(\frac{1}{d},\ldots,\frac{1}{d}\right) &\prec \left(\frac{1}{d-1},\ldots,\frac{1}{d-1},0\right)\prec\ldots\\ &\prec \left(\frac{1}{2},\frac{1}{2},0,\ldots,0\right)\prec(1,0,\ldots,0).\end{aligned}$$

T-transforms ("Robin Hood transfers")

A T-transform is a linear transformation whose matrix representation acts as the identity on all but 2 matrix components. On those two components it has the form:

$$T = \begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix},$$

where $0 \leq t \leq 1$.

Majorization: alternate definitions

The following statements are equivalent:

1. x is majorized by y;
2. $x = T_1 \cdots T_r y$, where $T_1, \ldots, T_r$ are T-transforms and $r < d$;
3. $x = My$ for some doubly stochastic matrix M (entries are probabilities; row sums and column sums are all 1)
   i.e. $x_i = \sum_j M_{ij} y_j \quad \forall i$

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The following statements are equivalent:

1. x is majorized by y;
2. $\sum_{i=1}^{d} \phi(x_i) \leq \sum_{i=1}^{d} \phi(y_i)$ for all continuous convex functions $\phi : \mathbb{R} \to \mathbb{R}$;

Note: A function $f : \mathbb{R}^d \to \mathbb{R}$ is *Schur-convex* if

$$x \prec y \Rightarrow f(x) \leq f(y).$$

Ex: $f(x) = x \log x$ is convex, so Shannon entropy $H(x) = -\sum_i x_i \log x_i$ (a measure of uncertainty used in information theory) is Schur-concave. This yields the following theorem:

Theorem

Let x and y be probability distributions such that $x \prec y$. Then $H(x) \geq H(y)$.

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Von Neumann entropy

Definition

Let ρ be a density matrix. Then we define the Von Neumann entropy of ρ as follows $S(\rho) = -Tr(\rho \log(\rho))$.

The Von Neumann entropy of a density matrix is the Shannon entropy of its eigenvalues. Hence if $\lambda_{\rho'} \prec \lambda_{\rho}$, then $S(\rho') \geq S(\rho)$.

Majorization: alternate definitions (cont.)

Theorem (Rado's theorem)

x is majorized by y iff x lies in the convex hull of vectors Py, where P is any permutation matrix (that is, x is contained in the convex hull of $(y_{\sigma(1)}, \ldots, y_{\sigma(d)})$, where σ is any permutation on d elements)

Random unitary channels

Definition

A quantum channel Φ from $B(\mathcal{H})$ to $B(\mathcal{H})$ is said to be random unitary channel if there exists a set of nonnegative reals $\{p_i\}_{i=1}^k$ which sum to one and set of unitaries $\{U_i\}_{i=1}^k$ such that $\Phi(\rho) = \sum_{i=1}^k p_i U_i^* \rho U_i$.

Note: $\{$Random unitary channels$\} \subseteq \{$Unital quantum channels$\}$ (with equality iff dimension of $\mathcal{H}$ is two which is the 1 qubit case).

Uhlmann

Theorem (Uhlmann)

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let ρ and ρ' be density matrices in $B(\mathcal{H})$. Then the following are equivalent:

1. *The vector of eigenvalues of ρ' is majorized by the vector of eigenvalues of ρ. (i.e. $\lambda_{\rho'} \prec \lambda_\rho$).*
2. *There exists a random unitary channel $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\Phi(\rho) = \rho'$*
3. *There exists a unital quantum channel $\Psi : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\Psi(\rho) = \rho'$*

Horn

A matrix $U = (u_{ij})$ is *orthostochastic* if there exists an orthogonal matrix $V = (v_{ij})$ such that $u_{ij} = v_{ij}^2$. Orthostochastic matrices are necessarily doubly stochastic. An example of a doubly stochastic matrix that is *not* orthostochastic is (must be at least 3×3):

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Lemma (Horn)

Let $x, y \in \mathbb{R}^n$. Then $x \prec y$ iff there exists an n by n orthostochastic matrix U such that $x = Uy$.

Corollary (Schur-Horn)

Let $x, y \in \mathbb{R}^n$. Then $x \prec y$ iff there exists an n by n Hermitian matrix H with $h_{ii} = x_i$ and $\lambda_i(H) = y_i$ for all i.

LOCC Explained

If two parties, Alice and Bob, can only carry out operations on their local systems and have a classical communication channel to transmit bits, it is called LOCC.

local operation : trace decreasing CP map,

$$\Phi_A \otimes \Phi_B : B(\mathcal{H}_A \otimes \mathcal{H}_B) \to B(\mathcal{H}'_A \otimes \mathcal{H}'_B)$$

that acts separately on each component of the tensor product

$$\Phi_A \otimes \Phi_B = (\Phi_A \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \Phi_B)$$

LOCC Explained

We shall call $\mathcal{H}_A$ Alice's system and $\mathcal{H}_B$ Bob's system in what follows.
Let $\text{diag}(\mathbb{M}_d) \subset \mathbb{M}_d$ denote classical algebra of diagonal matrices in some $\mathbb{M}_d$.
Classical communication is mathematically represented by

$$\Phi_A : B(\mathcal{H}_A) \to B(\mathcal{H}'_A) \otimes \text{diag}(\mathbb{M}_d)$$

and/or

$$\Phi_B : B(\mathcal{H}_B) \otimes \text{diag}(\mathbb{M}_d) \to B(\mathcal{H}'_B)$$

Notation

$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a pure state (unit vector), often identified with its corresponding rank-one projection $|\psi\rangle\langle\psi| \in B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$.
We denote by $\rho_\psi \equiv \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ the state of Alice's system, and λ_ψ the vector of eigenvalues of ρ_ψ.

Schmidt Decomposition

Any pure state $|x\rangle$ of a composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ may be written in the form $|x\rangle = \sum_i \sqrt{\lambda_i}|i_A\rangle|i_B\rangle$, where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, and $|i_A\rangle$ ($|i_B\rangle$) form an orthonormal basis for system $\mathcal{H}_A$ ($\mathcal{H}_B$). Note that ρ_x has eigenvalues λ_i.

Nielsen's Theorem

We say that $|\psi\rangle \to |\phi\rangle$, read "$|\psi\rangle$ transforms to $|\phi\rangle$" if $|\psi\rangle$ can be transformed into $|\phi\rangle$ by local operations and potentially unlimited two-way classical communication.

Theorem (Nielsen, 1999)

We can transform $|\psi\rangle$ to $|\phi\rangle$ using local operations and classical communication if and only if λ_ψ is majorized by λ_ϕ. That is,

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Physical Meaning

This theorem is significant because any entangled state can be transformed via LOCC into a state that is less or equally entangled. Thus this theorem states that $|\psi\rangle$ is at least as entangled as $|\phi\rangle$ if and only if the eigenvalues of $\text{tr}_B(|\psi\rangle\langle\psi|)$ are majorized by the eigenvalues of $\text{tr}_B(|\phi\rangle\langle\phi|)$.

EPR Paradox

Entangled states violate the classical principle of locality—the idea that an object is directly influenced only by its immediate surroundings.

The easiest way to see evidence of entanglement is to measure one component of an entangled state. This measurement fixes the value of the other component of the state, implying non-local communication between the two parts.

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Example of Entanglement

Electron spin, when measured, can be either of two states:
spin up $\uparrow$ or spin down $\downarrow$.

In the absence of measurement, electron spin is in a *superposition* of the two states $\uparrow / \downarrow$ (even without entanglement).

Consider an entangled electron pair; suppose the state of one of the electrons is measured as "spin up". This measurement fixes the state of the other electron—it will be "spin down".

In this sense, even when spatially separated, entangled electron pairs behave as a single quantum object.

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A Generalization

Theorem (Vidal 1999)

We can transform $|\psi\rangle$ to $|\phi\rangle$ with probability p using local operations and classical communication if and only if λ_ψ is super-majorized by $p\lambda_\phi$. That is,

$$|\psi\rangle \to |\phi\rangle \text{ iff } \lambda_\psi \prec^w p\lambda_\phi$$

Trumping

It is often the case that two vectors x and y are incomparable. *Trumping* (Jonathan-Plenio, 1999) is a generalization of majorization: it is sometimes possible to find a unit vector c, referred to as a "catalyst", such that $x \nprec y$ but $x \otimes c \prec y \otimes c$.

Definition

Let $x = (x_1, x_2, ..., x_d), y = (y_1, y_2, ..., y_d) \in \mathbb{R}^d$. We say that x is *trumped* by y and write $x \prec_T y$ if there exists a unit vector $c \in \mathbb{R}^n$ with positive components such that $x \otimes c \prec y \otimes c$.

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Trumping

Example (Jonathan-Plenio, 1999)

Let $x = (0.4, 0.4, 0.1, 0.1)$ and $y = (0.5, 0.25, 0.25, 0)$. x is not majorized by y (indeed, $0.4 < 0.5$ but $0.4 + 0.4 > 0.5 + 0.25$).
If $c = (0.6, 0.4)$, then $x \otimes c = (0.24, 0.24, 0.16, 0.16, 0.06, 0.06, 0.04, 0.04)$ and $y \otimes c = (0.30, 0.20, 0.15, 0.15, 0.1, 0.1, 0, 0)$.
One can check that $x \otimes c$ is majorized by $y \otimes c$, so x is trumped by y.

Trumping

It was shown (Daftuar-Klimesh, 2001) that the dimension of the catalyst may be arbitrarily large.

Note that we can assume without loss of generality that at least one of the vectors x or y has no zero components.

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Aubrun-Nechita

We can consider vectors to be infinite-dimensional by appending 0's

Let $T_{<\infty}(y)$ be the set of all probability vectors x having finite support, such that $x \prec_T y$.

Let $M_{<\infty}(y)$ be the set of all probability vectors x having finite support, such that $x^{\otimes n} \prec y^{\otimes n}$.

Note $M_d(y) \subseteq T_d(y)$, where d is the dimension of x.

Theorem (Aubrun-Nechita, 2008)

Let x and y be two probability vectors of finite support. The following are equivalent:

1. $x \in \overline{T_{<\infty}(y)}$;
2. $x \in \overline{M_{<\infty}(y)}$;
3. $||x||_p \leq ||y||_p$ *for all* $p \geq 1$,

where the closure is wrt the ℓ_1 *norm.*

Set up for Turgut's Thm

Define $S(x) = -\sum_{i=1}^{d} x_i \log x_i$, which we recall is the formula for the von Neumann entropy of a density matrix with eigenvalues x_i.

Define $A_\nu(x) = \left(\frac{1}{d}\sum_{i=1}^{d} x_i^\nu\right)^{\frac{1}{\nu}}$ for real numbers $\nu \neq 0$ and $A_0(x) = \left(\prod_{i=1}^{d} x_i\right)^{\frac{1}{d}}$.

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Turgut's Theorem

Theorem (Turgut, 2007)

For two real d-dimensional vectors x and y with non-negative components such that x has non-zero elements and the vectors are distinct up to permutation (i.e. $x^{\uparrow} \neq y^{\uparrow}$), the relation $x \prec_T y$ is equivalent to the following three strict inequalities:

1. $A_\nu(x) > A_\nu(y), \quad \forall \nu \in (-\infty, 1),$
2. $A_\nu(x) < A_\nu(y), \quad \forall \nu \in (1, \infty),$
3. $S(x) > S(y).$

Klimesh's Theorem

Klimesh (2004, 2007) establishes a theorem that shows trumping is equivalent to a series of inequalities for a family of additive Schur-convex functions. For a d-dimensional probability vector x, let

$$f_r(x) = \begin{cases} \ln \sum_{i=1}^d x_i^r & (r > 1); \\ \sum_{i=1}^d x_i \ln x_i & (r = 1); \\ -\ln \sum_{i=1}^d x_i^r & (0 < r < 1); \\ -\sum_{i=1}^d \ln x_i & (r = 0); \\ \ln \sum_{i=1}^d x_i^r & (r < 0). \end{cases}$$

If any of the components of x are 0, we take $f_r(x) = \infty$ for $r \leq 0$.

Theorem (Klimesh, 2004/07)

Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be d-dimensional probability vectors. Suppose that x and y do not both contain components equal to 0 and that $x^\uparrow \neq y^\uparrow$. Then $x \prec_T y$ if and only if $f_r(x) < f_r(y)$ for all real numbers r.

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Power Majorization

Power majorization is a more refined notion of majorization.

Definition

Let x and y be vectors of non-negative components. We say that x is *power majorized* by y, denoted $x \preceq_p y$, if $x_1^p + \cdots + x_d^p \leq y_1^p + \cdots + y_d^p$ for all $p \geq 1, p \leq 0$ and the inequality switches direction when $0 \leq p \leq 1$. We define *strict power majorization*, denoted $x \prec_p y$, to be power majorization with strict inequality, and equality if and only if $p = 0, 1$.

Proposition (Kribs-Pereira-P.)

Power majorization can be expressed in terms of Klimesh's functionals: Let x and y be vectors in $\mathbb{R}^d$ with positive components. Then $x \preceq_p y$ if and only if $f_r(x) \leq f_r(y)$ for all $r \in \mathbb{R}$.

Proposition (Kribs-Pereira-P.)

Let x and y be vectors in $\mathbb{R}^d$ with positive components with $x \prec_p y$, then $x \prec_T y$ provided that $\prod_{i=1}^d x_i \neq \prod_{i=1}^d y_i$ and $\prod_{i=1}^d x_i^{x_i} \neq \prod_{i=1}^d y_i^{y_i}$.

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Some Sets of Interest

Let $S(y) = \{x \in (0,\infty)^d : x \prec y\}$, $T(y) = \{x \in (0,\infty)^d : x \prec_T y\}$, and $P(y) = \{x \in (0,\infty)^d : x \preceq_p y\}$.

While the geometric properties of $S(y)$ are quite well-known; there is less known about $T(y)$ and even less known about $P(y)$. It is clear that $S(y) \subseteq T(y) \subseteq P(y)$.

We begin with the following closure relation.

Theorem (Kribs-Pereira-P.)

Let $y \in \mathbb{R}^d$ all of whose components are positive, then the set $P(y) = \overline{T(y)}$.

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We know that $T(y)$ is convex (see, e.g. Daftuar & Klimesh's work). Since $P(y) = \overline{T(y)}$, it follows that $P(y)$ is a convex set.

Thus the set $P(y)$ is a closed convex set, and so it is the convex hull of its extreme points.

Rado's theorem (majorization): $x \prec y$ if and only if $(x_1, \ldots, x_d)$ is contained in the convex hull of $(y_{\sigma(1)}, \ldots, y_{\sigma(d)})$, where σ is any permutation on d elements.

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Theorem (Kribs-Pereira-P.)

Let $y \in \mathbb{R}^d$ all of whose components are positive and let $x \in P(y)$. Then the following are equivalent:

1. *x is an extreme point of $P(y)$.*
2. *$f_r(x) = f_r(y)$ for some $r \in \mathbb{R}$.*
3. *Either x is not trumped by y or there exists some d by d permutation matrix P such that $x = Py$.*

Examples of Trumping

A lower bound on the dimension of the catalyst was found by Sanders and Gour (2009) based on Generalised concurrence monotones.
Two states ψ and ϕ with Schmidt vectors

$$\begin{aligned}\sigma(\psi) &= \left(\frac{19}{351}, \frac{1}{13}, \frac{64}{351}, \frac{71}{351}, \frac{3}{13}, \frac{89}{351}\right)\\ \sigma(\phi) &= \left(\frac{9}{196}, \frac{25}{196}, \frac{13}{98}, \frac{5}{28}, \frac{3}{14}, \frac{59}{196}\right).\end{aligned}$$

There is no LOCC transformation between these two states (no majorization); one can verify numerically that there indeed exists a catalyst; their results show that such a catalyst must be of dimension at least 3 here.

Example

A system first considered by Bennett is

$$\frac{1^p}{1^{p+1}} < \frac{1^p + 3^p}{2^{p+1}} < \frac{1^p + 3^p + 5^p}{3^{p+1}} < \cdots \frac{1^p + 3^p + \cdots + (2n-1)^p}{n^{p+1}} < \cdots \quad (1)$$

for $p > 1, p < 0$ and reversed for $0 < p < 1$.
This system leads to power majorization: consider e.g. the second inequality and cross-multiply:

$$3^p + 3^p + 3^p + 9^p + 9^p + 9^p \leq 2^p + 2^p + 6^p + 6^p + 10^p + 10^p$$

(for appropriate p). In other words,
$x = (3, 3, 3, 9, 9, 9) \preceq_p (2, 2, 6, 6, 10, 10) = y$. We can in fact use the observations above to show that we have $x \prec_T y$. That is, this system gives us an infinite sequence of pairs of vectors (x, y) where x is trumped (but not majorized) by y.

Infinite-Dimensional Majorization

Definition

ℓ_1^+ consists of all $x = \{x_n\}_{n=1}^{\infty} \in \ell_1(\mathbb{R})$ with the property that $x_n \geq 0$ for all $n \in \mathbb{N}$ and exactly one of the sets $\{n \in \mathbb{N} : x_n > 0\}$ and $\{n \in \mathbb{N} : x_n = 0\}$ is finite.

Definition

For any $x, y \in \ell_1^+$, we say that x is majorized by y, written $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad k \in \mathbb{N}$$

$$\text{and} \sum_{i=1}^{\infty} x_i^{\downarrow} = \sum_{i=1}^{\infty} y_i^{\downarrow}.$$

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Infinite-Dimensional Setting

Definition (Owari et al, 2008)

Let $|\psi\rangle$ and $|\phi\rangle$ be unit vectors (states) in $\mathcal{H}_A \otimes \mathcal{H}_B$. We say that $|\psi\rangle$ is *ϵ-convertible* to $|\phi\rangle$ by LOCC if for any $\epsilon > 0$, there exists an LOCC operation Λ satisfying $||\Lambda(|\psi\rangle\langle\psi|) - |\phi\rangle\langle\phi|||_{\mathrm{Tr}} < \epsilon$, where $||\cdot||_{\mathrm{Tr}}$ is the trace norm.

The concept of ϵ-convertibility allows for the extension of Nielsen's theorem, which gives necessary and sufficient conditions for LOCC transformations, to the infinite-dimensional setting:

Theorem (Owari et al, 2008)

$|\psi\rangle$ is ϵ-convertible to $|\phi\rangle$ by LOCC if and only if $\lambda \prec \mu$, where λ and μ are the vectors of Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively.

Infinite-Dimensional Setting

Nielsen's result has been extended to vectors in ℓ_1^+ by way of ϵ-convertibility for LOCC by Owari et. al. (2008). We can thus define infinite-dimensional trumping:

Definition

For any $x, y \in \ell_1^+$, we say that x is *trumped* by y, written $x \prec_T y$, if there exists a unit vector $c \in \ell_1^+$ with all positive components such that $x \otimes c \prec y \otimes c$.

The *catalyst* c is allowed to have infinite length in this setting, though it may be the case that it has finite length.

Questions

- It has been shown that the catalyst c may have arbitrary length. That is, the dimension of c does not depend on x or y, and if we know $x \prec_T y$, we don't immediately know the dimension of c. In a real-world physical application, we may be forced to use only 2-dimensional c. If we require $c \in \mathbb{R}^2$, can we say something about all x trumped by y?
- Can we characterise MLOCC? (multiple copies of the states, which allows for LOCC transformation): $x^{\otimes n} \prec y^{\otimes n}$ (Note that this implies trumping with catalyst

$$c = \frac{1}{n}(x^{\otimes n} \oplus x^{\otimes (n-1)} \otimes y \oplus \cdots \oplus y^{\otimes n})$$

- Can we extend trumping and MLOCC in terms of operators in finite von Neumann algebras?
- Schur-Horn or Rado theorems for trumping/MLOCC/power majorization in II_∞ factors?